22. On the Acyclicity of Free Cobar Constructions. II

By Kazuhiko AOMOTO

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1977)

Examle 1. Let M be a connected simplicial complex and x_0 be a base point of M. Let N be a maximal tree in M containing x_0. We denote by $C_*(M/N)=\oplus_{p=0}^{\infty} C_p(M/N)$ the CW-complex with the only one vertex x_0. Let G be the edge path group of M/N with the base point x_0. The set of all reduced closed pathes in M/N with the base point x_0 forms a free group F. Let L be the two-sided ideal of $Z[F]$, generated by the elements of the form

$$\delta_1 T\langle v_0, v_1, v_2\rangle = \widetilde{T\langle v_0, v_1\rangle}\cdot\widetilde{T\langle v_1, v_2\rangle} - \widetilde{T\langle v_0, v_2\rangle} \tag{1.1}$$

for any reduced 2-simplex T of M/N: $\langle v_0, v_1, v_2\rangle \to M/N$, where $\widetilde{T\langle v_i, v_j\rangle} \in F$. We define X^f as $Z[F]\otimes C_*(M/N)$. For a reduced n-simplex T regarded as an element of X_n^f or A_{n-1} the formulae for ∂_n^f or δ_{n-1} are given as follows respectively:

$$\begin{aligned} &\partial_n^f T\langle v_0, v_1, \cdots, v_n\rangle = \widetilde{T\langle v_0, v_1\rangle}\cdot T\langle v_1, \cdots, v_n\rangle \\ &\quad + \textstyle\sum_{i=1}^{n} (-1)^i \cdot T\langle v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n\rangle, \quad n\geqq 2 \quad \text{and} \\ &\partial_1^f T\langle v_0, v_1\rangle = \widetilde{T\langle v_0, v_1\rangle} - 1 \in Z[F]^+, \quad n=1, \end{aligned} \tag{1.2}$$

where $\widetilde{T\langle v_0, v_1\rangle}$ lies in F, and

$$\begin{aligned} &\delta_{n-1} T\langle v_0, v_1, \cdots, v_n\rangle = \widetilde{T\langle v_0, v_1\rangle}\cdot T\langle v_1, \cdots, v_n\rangle \\ &\quad + \textstyle\sum_{i=1}^{n-1} (-1)^i \cdot T\langle v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n\rangle \\ &\quad + (-1)^n \cdot T\langle v_0, v_1, \cdots, v_{n-1}\rangle \cdot \widetilde{T\langle v_{n-1}, v_n\rangle} \\ &\quad + \textstyle\sum_{i=2}^{n-2} (-1)^{i-1} \cdot T\langle v_0, v_1, \cdots, v_i\rangle \cdot T\langle v_i, \cdots, v_n\rangle, \quad n\geqq 3, \end{aligned} \tag{1.3}$$

where $\widetilde{T\langle v_0, v_1\rangle}$ and $\widetilde{T\langle v_{n-1}, v_n\rangle}$, $n\geqq 3$, $\widetilde{T\langle v_0, v_1\rangle}$, $\widetilde{T\langle v_1, v_2\rangle}$ and $\widetilde{T\langle v_0, v_2\rangle}$, $n=2$ are in F. The free cobar construction (A, δ) thus defined satisfies Assumptions 1 and 2 in [3]. This is nothing but a modification of Adams cobar construction. So by [3]

Theorem 1. *$\mathrm{H}_p(A)\cong(0)$, $p\geqq 1$ and $\mathrm{H}_0(A)\cong Z[G]$ if and only if M is a $\mathrm{K}(\Pi, 1)$ space.*

This is also related with Pfeiffer-Smith-Whitehead identity relations [4].

Example 2. Let $\mathfrak{G}$ be a Lie algebra over Z and $\mathcal{E}(\mathfrak{G})$ or $T(\mathfrak{G})$ be its envelopping algebra or tensor algebra respectively. We consider the normalized standard complex (X, ∂) on $\mathcal{E}(\mathfrak{G})$, $X=\mathcal{E}(\mathfrak{G})\otimes\Lambda^*\mathfrak{G}$, where $\Lambda^*(\mathfrak{G})$ denotes the exterior algebra of $\mathfrak{G}$. We put $X^f = T(\mathfrak{G})\otimes\Lambda^*\mathfrak{G}$ and define for a $x_1\wedge x_2\wedge\cdots\wedge x_n \in X_n^f$, $x_j\in\mathfrak{G}$, $1\leqq j\leqq n$,

$$\begin{aligned}(2.1)\qquad &\partial_n^f(x_1\wedge x_2\wedge\cdots\wedge x_n)\\ &=\textstyle\sum_{i=1}^n(-1)^i\cdot x_i\otimes(x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_n)\\ &\quad+\textstyle\sum_{1\leqq i<j\leqq n}(-1)^{i+j}[x_i,x_j]\wedge x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\\ &\qquad\qquad\wedge x_{j-1}\wedge x_{j+1}\cdots\wedge x_n.\end{aligned}$$

As an element of A_{n-1} we define

$$\begin{aligned}(2.2)\qquad &\delta_{n-1}(x_1\wedge\cdots\wedge x_n)\\ &=\textstyle\sum_{i=1}^n(-1)^i x_i\cdot(x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_n)\\ &\quad+\textstyle\sum_{1\leqq i<j\leqq n}(-1)^{i+j}([x_i,x_j]\wedge x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\\ &\qquad\qquad\wedge x_{j-1}\wedge x_{j+1}\wedge\cdots\wedge x_n)\\ &\quad+\textstyle\sum_{\nu=2}^{n-2}\sum_{\substack{i_1<\cdots<i_\nu\\ i_{\nu+1}<\cdots<i_n}}(-1)^{\nu-1}\operatorname{sgn}\binom{1\ 2\cdots n}{i_1 i_2\cdots i_n}(x_{i_1}\wedge\cdots\wedge x_{i_\nu})(x_{i_{\nu+1}}\\ &\qquad\qquad\wedge\cdots\wedge x_{i_n})\\ &\quad+\textstyle\sum_{i=1}^n(-1)^i(x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_n)\cdot x_i,\quad n\geqq 3\end{aligned}$$

and

$$\delta_1(x_1\wedge x_2)=x_1\cdot x_2-x_2\cdot x_1-[x_1,x_2],\quad n=2.$$

Then (A,δ) satisfies Assumptions 1 and 2 in [3]. Consequently

Theorem 2. $\mathrm{H}_p(A)\cong(0)$, $p\geqq 1$ *and* $\mathrm{H}_0(A)\cong\mathcal{E}(\mathfrak{G})$, $p=0$.

Example 3. Let M be the complement $\boldsymbol{P}^n-S$ of the union of hyperplanes $S_j: f_j=0$, $S=\bigcup_{j=1}^{m+1}S_j$ in the complex projective space $\boldsymbol{P}^n$, where we put S_{m+1} to be the hyperplane at infinity. The holonomy Lie algebra $\mathfrak{G}(S)$ attached to the configuration S is defined to be the Lie algebra generated by the symbols $x_1, x_2, \cdots, x_m$ with the defining relations (see [1])

$$(3.1)\qquad (\textstyle\sum_{j=1}^m x_j\cdot df_j/f_j)\wedge(\sum_{j=1}^m x_j\cdot df_j/f_j)=0.$$

Let $\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S)$ be the space of logarithmic forms along S which is known to be generated by $\mathcal{D}^1(\boldsymbol{P}^n,\log S)$ (see for example [2] p. 292). Let X be $\mathrm{Hom}_{\mathcal{E}(\mathfrak{G})}(\mathcal{E}(\mathfrak{G})\otimes\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S),\ \mathcal{E}(\mathfrak{G}))$, then X becomes a complex defined by

$$(3.2)\qquad \partial\lambda(\varphi)=\lambda(\textstyle\sum_{j=1}^m x_j(df_j/f_j)\wedge\varphi)=\sum_{j=1}^m x_j\cdot\lambda((df_j/f_j)\wedge\varphi)$$

for $\lambda\in X$, $\varphi\in\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S)$. Let $\mathrm{B}(\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S))$ be the reduced bar construction on $\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S)$, i.e. the Chen complex consisting of iterated integrals of $\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S)$. $\mathfrak{G}$ being a graded Lie algebra, we have from [3]

Theorem 3. (X,∂) *is acyclic if and only if* $H^p(\mathrm{B}(\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S)))$ *vanishes for* $p\geqq 1$. *In this case* $H^0(\mathrm{B}(\mathcal{D}^\cdot(\boldsymbol{P}^n,\log S)))$, *the space of hyperlogarithmic functions, is isomorphic as graded algebra, to the dual of* $\widecheck{\mathcal{E}(\mathfrak{G})}$, *the completion of* $\mathcal{E}(\mathfrak{G})$ *with respect to the augmentation ideal* $\mathcal{E}(\mathfrak{G})^+$.

It is conjectured that *the complex* (X,∂) *is acyclic if and only if* M *is a* $\mathrm{K}(\Pi,1)$ *space*. In particular let S consist of 5 lines $z_1=0,1$, $z_2=0,1$ and $z_1=z_2$ in the 2-dimensional complex affine space $\boldsymbol{C}^2$. Then $\boldsymbol{C}^2-S$ is $\mathrm{K}(\Pi,1)$ and (X,∂) is acyclic. We take $\omega_{01}=dz_1/z_1$, $\omega_{02}=dz_2/z_2$, $\omega_{12}=d(z_1-z_2)/(z_1-z_2)$, $\omega_{13}=dz_1/(z_1-1)$, $\omega_{23}=dz_2/(z_2-1)$ and $\omega_{03}=0$, with

$\omega_{ij}=\omega_{ji}$. Then the iterated integral

$$(3.3)\qquad \Phi[ijk]=\int \omega_{ij}\omega_{jk}+\omega_{jk}\omega_{ki}+\omega_{ki}\omega_{ij}-\omega_{jk}\omega_{ij}-\omega_{ki}\omega_{jk}-\omega_{ij}\omega_{ki}$$

is closed in $B^0(\mathcal{D}^\cdot(\boldsymbol{P}^2, \log S))$ which defines a *bilogarithmic function.* The function $\sum_{\nu=1}^{4}(-1)^\nu\cdot\Phi[1\cdot\overset{\nu}{\vee}\cdot 4]$ was used in the combinatorial formula for the 1st Pontrajagin class in [5].

References

[1] K. Aomoto: Sur la forme de connexion euclidienne du type régulier. Proc. Japan Acad., **46**, 660–662 (1970).

[2] ——: Les équations aux différences linéaires et les intégrales des fonctions multiformes. Jour. of The Fac. of Sci. Univ. of Tokyo, **22**, 271–297 (1975).

[3] ——: On the acyclicity of free cobar constructions I. Proc. Japan Acad., **53A**, 35–36 (1977).

[4] W. T. Van Est: Pfeiffer-Smith-Whitehead non-commutative homology and cohomology (1977) (preprint).

[5] A. M. Gabrielov, I. M. Gelfand, and M. V. Losik: Combinatorial calculation of characteristic classes. Funct. Anal. and Its Appli., **9**, 5–26 (1975) (in Russian).