20. On Multivalent Functions in Multiply Connected Domains. II

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1. Introduction. In the preceding paper [1] we extended Rengel's results ([4] or [3]) to the case of circumferentially mean p-valent functions. In this paper we shall treat the case of areally mean p-valent functions defined as follows.

Let $n(R, \Phi)$ denote the number of roots of the equation $f(z)=w=\mathrm{Re}^{i\Phi}$ in a domain D. If for a certain positive integer p,

$$\int_0^R \left(\int_0^{2\pi} n(R, \Phi)d\Phi\right) RdR \leq p\pi R^2 \qquad (0 \leq R < \infty), \tag{1.1}$$

then $f(z)$ is called to be areally mean p-valent (cf. [2]).

As defined in [1], D_1, D_2, D_3, D_4, D_5 and D_6 denote the n-ply connected, representative domains of the following types respectively.

D_1: an annulus, $(0<)\ r_1<|z|<r_2\ (<\infty)$ with $(n-2)$ circular arc slits centered at the origin.

D_2: an annulus, $(0<)\ r_1<|z|<r_2\ (<\infty)$ with $(n-2)$ radial slits emanating from the origin.

D_3: the unit circle with $(n-1)$ circular arc slits centered at the origin.

D_4: the unit circle with $(n-1)$ radial slits emanating from the origin.

D_5: the whole plane with n circular arc slits centered at the origin.

D_6: the whole plane with n radial slits emanating from the origin.

2. We shall first quote Hayman's result (p. 33 in [2]).

Lemma. *Let $f(z)=\mathrm{Re}^{i\Phi}$ be single-valued, regular, areally mean p-valent in a domain D and $n(R, \Phi)$ denote the quantity defined above. Let $R_1=\inf_{z\in D}|f(z)|$ and $R_2=\sup_{z\in D}|f(z)|$. Then we have*

$$\int_{R_1}^{R_2} \frac{p(R)}{R}dR \leq p\left(\log \frac{R_2}{R_1}+\frac{1}{2}\right) \tag{2.1}$$
$$\left(p(R)\equiv \frac{1}{2\pi}\int_0^{2\pi} n(R, \Phi)d\Phi\right).$$

Hereafter we shall derive the results in this paper by the method quite similar to [1].

Theorem 2.1. *Let $f(z)$ be single-valued, regular, areally mean p-valent in D_1 and satisfy the condition*

$$\int_C |d \arg f(z)| \geq 2\pi p \qquad (C: |z|=r \ (r_1<r<r_2)),$$

where the circle C does not contain any circular slit of D_1. Then we have the following inequality:

$$p \log \frac{r_2}{r_1} \leq \log \frac{R_2}{R_1} + \frac{1}{2} \qquad \Big(R_1 \equiv \inf_{z \in D_1} |f(z)|,\ R_2 \equiv \sup_{z \in D_1} |f(z)|\Big). \tag{2.2}$$

Proof. As shown in [1],

$$\iint_{D_1} \rho^2 r dr d\varphi \geq \frac{p^2}{2\pi} \log \frac{r_2}{r_1} \qquad \Big(\rho \equiv \frac{1}{2\pi} \Big|\frac{f'(z)}{f(z)}\Big|,\ z=re^{i\varphi}\Big), \tag{2.3}$$

$$(2\pi)^2 \iint_{D_1} \rho^2 r dr d\varphi = \iint_{D_1^*} \frac{n(R, \Phi)}{R} dR d\Phi \tag{2.4}$$

($z=re^{i\varphi}$, $w=\mathrm{Re}^{i\Phi}$, $D_1^*=$the image domain of D_1).

On the other hand

$$\iint_{D_1^*} \frac{n(R, \Phi)}{R} dR d\Phi = \int_0^{2\pi} \int_{R_1}^{R_2} \frac{n(R, \Phi)}{R} dR d\Phi = 2\pi \int_{R_1}^{R_2} \frac{p(R)}{R} dR.$$

Therefore, by means of Lemma we have

$$\frac{p^2}{2\pi} \log \frac{r_2}{r_1} \leq \frac{p}{2\pi}\Big(\log \frac{R_2}{R_1} + \frac{1}{2}\Big). \tag{2.5}$$

Theorem 2.2. *Let $f(z)$ be single-valued, regular, and areally mean p-valent in D_2. Let $M=\{\gamma_\varphi\}$ denote the family of the segments $r_1<|z|<r_2$, $\arg z=\varphi$ $(0\leq\varphi<2\pi)$ which do not contain any radial slit of D_2. Then we have the following inequality.*

$$p\Big(\log \frac{R_2}{R_1} + \frac{1}{2}\Big) \log \frac{r_2}{r_1} \geq A^2 \tag{2.6}$$

$$\Big(A \equiv \inf_{\gamma_\varphi \in M} \int_{r_1}^{r_2} \Big|\frac{f'(z)}{f(z)}\Big| dr,\ R_1 \equiv \inf_{z \in D_2} |f(z)|,\ R_2 \equiv \sup_{z \in D_2} |f(z)|\Big).$$

Proof. As shown in [1],

$$\iint_{D_2} \rho^2 r dr d\varphi \geq \frac{A^2}{2\pi \log (r_2/r_1)} \qquad \Big(\rho = \frac{1}{2\pi} \Big|\frac{f'(z)}{f(z)}\Big|\Big). \tag{2.7}$$

On the other hand, by means of Lemma, we have

$$\iint_{D_2} \rho^2 r dr d\varphi \leq \frac{p}{2\pi}\Big(\log \frac{R_2}{R_1} + \frac{1}{2}\Big). \tag{2.8}$$

3. Next we shall show some applications of Theorem 2.1 and Theorem 2.2.

Theorem 3.1. *Let $f(z)$ be single-valued, regular, areally mean p-valent, and bounded, that is, $|f(z)|<1$ in D_3. Moreover let*

$$\int_{\gamma_\nu} d \arg f(z) = 0 \qquad (\nu=1, 2, \cdots, n-1) \tag{3.1}$$

along every curve γ_ν in D_3 which is sufficiently near to the slit S_ν ($\nu=1, 2, \cdots, n-1$) and encloses it simply, and $f(z)$ be expanded in the neighborhood of the origin as follows:

$$f(z)=a_p z^p + a_{p+1} z^{p+1} + \cdots.$$

Then we have

$$|a_p| \leq e^{1/2}. \tag{3.2}$$

Proof. Let $\delta(\varepsilon)$ denote the nearest distance from the origin to the image of a small circle $|z|=\varepsilon$ by $w=f(z)$. Then we have

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon^p} = |a_p|. \tag{3.3}$$

By means of the same reasoning as shown in [1] and Theorem 2.1, we have

$$p \log \frac{1}{\varepsilon} \leq \log \frac{1}{\delta(\varepsilon)} + \frac{1}{2}. \tag{3.4}$$

we can derive $|a_p| \leq e^{1/2}$ from (3.3) and (3.4).

Theorem 3.2. *Let $f(z)$ be single-valued, regular, areally mean p-valent and bounded, that is, $|f(z)|<1$ in D_4. Let, in a neighborhood of the origin,*

$$f(z)=a_p z^p + a_{p+1} z^{p+1} + \cdots.$$

Then we have

$$|a_p| \geq m^2 e^{-1/2} \qquad \left(m = \min_{|z|=1} |f(z)|\right). \tag{3.5}$$

Proof. Let $\delta(\varepsilon)$ or $\delta^*(\varepsilon)$ denote respectively the longest or nearest distance from the origin to the image of $|z|=\varepsilon$ by $w=f(z)$. Then, by means of the same reasoning as shown in [1] and Theorem 2.2, we have

$$\left(\log \frac{m}{\delta(\varepsilon)}\right)^2 \leq p\left(\log \frac{1}{\delta^*(\varepsilon)} + \frac{1}{2}\right) \log \frac{1}{\varepsilon}. \tag{3.6}$$

On the other hand

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon^p} = \lim_{\varepsilon\to 0} \frac{\delta^*(\varepsilon)}{\varepsilon^p} = |a_p|. \tag{3.7}$$

Hence, by letting ε tend to 0 and making use of (3.6) and (3.7), we have

$$0 \geq \log \frac{m^2}{|a_p|} - \frac{1}{2}. \tag{3.8}$$

4. Lastly we shall state the results similar to Theorem 3.1 and Theorem 3.2 in the cases of D_5 and D_6 which can be also proved by the method indicated in [1].

Theorem 4.1. *Let $f(z)$ be single-valued, regular, except for the pole at ∞, areally mean p-valent in D_5 and expanded in a neighborhood of the origin*

$$f(z)=z^p + a_{p+1} z^{p+1} + \cdots.$$

Moreover let

$$\int_{\gamma_\nu} d \arg f(z)=0 \qquad (\nu=1, 2, \cdots, n)$$

for every simply closed curve γ_ν which is sufficiently near to each circular arc slit S_ν and encloses S_ν simply. Then we have

$$\lim_{z\to\infty}\left|\frac{f(z)}{z^p}\right| \geq e^{-1/2}. \tag{4.1}$$

Theorem 4.2. *Let $f(z)$ be single-valued, regular, except at $z=\infty$, areally mean p-valent in D_6 and let in a neighborhood of $z=\infty$,*

$$f(z)=z^p \sum_{n=0}^{\infty} b_n z^{-n} \qquad (b_0=1).$$

Moreover let in a neighborhood of the origin

$$f(z)=a_p z^p + a_{p+1} z^{p+1} + \cdots.$$

Then we have

$$|a_p| \geq e^{-1/2}. \tag{4.2}$$

References

[1] H. Abe: On multivalent functions in multiply connected domains. I. Proc. Japan. Acad., **53**, 116–119 (1977).
[2] W. K. Hayman: Multivalent Functions. Cambridge tracts, No. 48 (1958).
[3] Y. Komatu: Conformal Mapping. II. Kyoritu tracts (Tokyo) (1949) (in Japanese).
[4] E. Rengel: Über einige Schlitztheoreme der konformen Abbildung. Schr. Math. Seminars. u. Inst. f. angew. Math., Univ. Berlin, **1**, 141–162 (1933).