# 18. *Weak Stability Implies Structural Stability under Axiom A*[*)]

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§1. **Introduction.** Let $\mathcal{D}^r(M)$ and $\mathfrak{X}^r(M)$ be the sets of diffeomorphisms and vector fields, respectively, of class $C^r$ on a smooth manifold $M$ with Whitney $C^r$ topology. $f \in \mathcal{D}^r(M)$ or $X \in \mathfrak{X}^r(M)$ is said to be *weakly $C^r$ stable* if and only if there is a neighborhood $N$ of $f$ or $X$ in $\mathcal{D}^r(M)$ or $\mathfrak{X}^r(M)$ such that for any $g$ or $Y$ in $N$ the set of all elements topologically equivalent to $g$ or $Y$ is dense in $N$, respectively.

In [2] and [3], weak stability researched. The set of all dynamical systems on $M$ which are weakly stable but not structurally stable is an open subset of $\mathcal{D}^r(M)$ or $\mathfrak{X}^r(M)$.

**Theorem 1** ([2]). *For $1 \leqq r \leqq \infty$ there exists an open set $N$ of $\mathfrak{X}^r(\boldsymbol{R}^2)$ such that* (i) *$N$ contains uncountably many equivalence classes of vector fields*; (ii) *for any $X$ in $N$ the set of all elements topologically equivalent to $X$ is dense in $N$, hence, any $X$ in $N$ is not structurally $C^r$ stable but is weakly $C^r$ stable*; *and* (iii) *any $X$ in $N$ is $C^r\Omega$-stable.*

**Theorem 2** ([3]). *Let $f$ be a $C^r$ diffeomorphism of a compact manifold, $1 \leqq r \leqq \infty$. If $f$ is weakly $C^r$ stable, then all periodic points of $f$ are hyperbolic.*

The following problem is open yet: Are there diffeomorphisms or flows on a compact manifold which are weakly stable but not structurally stable? Having in view of this problem, we ask when weak stability implies structural stability. This paper is motivated by this question. There is the following result about this question.

**Theorem 3** ([2]). *For a compact manifold $M$, let $f$ be a $C^r$ diffeomorphism on $M$, $r \geqq 1$. Suppose that $\Omega(f)$ is a finite set. Then, if $f$ is weakly $C^r$ stable $f$ is structurally $C^r$ stable.*

The main result of this paper is Theorem in the next section.

§2. **Result for $\mathcal{D}^r(M)$.** The following Proposition can be said to be an extension of Theorem 3, for Morse-Smale diffeomorphisms are structurally stable.

**Proposition.** *Let $M$ be compact and $1 \leqq r \leqq \infty$. Then, if $f \in \mathcal{D}^r(M)$ is weakly $C^r$ stable, $f$ is Kupka-Smale.*

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**Proof.** By Theorem 2 all periodic points are hyperbolic. Next, we show the transversality of $W^s(p, f)$ and $W^u(q, f)$, the stable and unstable manifolds of any periodic points $p$ and $q$. Suppose that there are periodic points $p, q$ and a point $x \in W^s(p, f) \cap W^u(q, f)$ such that $W^s(p, f)$ and $W^u(q, f)$ are not transverse at $x$. Since $f$ is weakly stable, there is an open neighborhood $N$ of $f$ in $\mathcal{D}^r(M)$ such that for any $f' \in N$ the set of all diffeomorphisms topologically equivalent to $f'$ is dense in $N$. Let the periods of the periodic points $p$ and $q$ are $m_p$ and $m_q$. By R. C. Robinson [5], there is a diffeomorphism $f_0$ in $N$ satisfying the following properties;

i) the above $p$ and $q$ are also the periodic points of $f_0$ of periods $m_p$ and $m_q$, respectively,

ii) the above $x$ is contained in $W^s(p, f_0)$ and $W^u(q, f_0)$, and

iii) the topological type of the intersection of $W^s(p, f_0)$ and $W^u(q, f_0)$ at $x$ is different from any transversal intersection.

On the other hand, there is a Kupka-Smale diffeomorphism $g_0$ in $N$. Put $\max\{m_p, m_q\} = m$. There is an open neighborhood $U$ of $g_0$ contained in $N$ such that any diffeomorphism $g$ in $U$ satisfies the following properties;

i) any periodic point of $g$ with period $\leqq m$ are hyperbolic,

ii) for any periodic points $p, q$ of $g$ with period $\leqq n$, $W^s(p, g)$ and $W^u(q, g)$ have transversal intersection.

Then, any $g$ in $U$ cannot topologically equivalent to $f_0$. Hence, the topological equivalence class of $f_0$ is never dense in $N$. Since this contradicts the definition of $N$, the stable and unstable manifolds of any periodic points of $f$ have transversal intersection. Therefore, $f$ is a Kupka-Smale diffeomorphism.

**Theorem.** *Let $f$ be an Axiom* A *$C^r$ diffeomorphism on a compact manifold $M$, $1 \leqq r \leqq \infty$. Then, if $f$ is weakly $C^r$ stable $f$ is structurally $C^r$ stable.*

**Proof.** Suppose that $f$ satisfies Axiom A and is weakly $C^r$ stable. By J. W. Robbin [4] and R. C. Robinson [6], Axiom A and strong transversality condition (ST condition) imply structural stability. So, we are sufficient if we prove that $f$ satisfies ST condition. There is an open neighbourhood $N$ of $f$ such that any $g$ in $N$ is weakly stable. Assume that $f$ does not satisfies ST condition, i.e., there are points $a, b$ in the non wandering set of $f$, $\Omega(f)$, such that $W^s(a, f)$ and $W^u(b, f)$ are never transversal at a point $x \in W^s(a, f) \cap W^u(b, f)$. If we can show that there is $g$ in $N$ such that there are periodic points $p, q$ of $g$ such that $W^s(p, g)$ and $W^u(q, g)$ intersect non transversally at a point $y$, then by Proposition $g$ is not weakly stable. But, since this contradicts the definition of $N, f$ satisfies ST condition. $g$ is found as follows. Since

$f$ satisfies Axiom A, using the result of M. W. Hirsch and C. C. Pugh [1] we see that for arbitrarily small neighborhood $U$ of the above $x$ there are periodic points $p$ and $q$ of $f$ near $a$ and $b$, respectively, such that $W^s(p,f)$ and $W^u(q,f)$ intersect with $U$ and that the pairs $W^s(a,f)$, $W^s(p,f)$ and $W^u(b,f)$, $W^u(q,f)$ are parallel in $U$. Then, by a perturbation of $f$ only on $f^{-1}(U)$, we can construct $g$ so that $W^s(p,g)$ and $W^u(q,g)$ intersect non transversally at a point $y$ in $U$. This proves Theorem.

**Remark.** Let $X \in \mathfrak{X}^r(M)$. When the non wandering set $\Omega(X)$ of $X$ consists of finite singular points and finite closed orbits the corresponding result for $\mathfrak{X}^r(M)$ to Theorem 3 is true. By the parallel method we can prove the corresponding results for $\mathfrak{X}^r(M)$ to Theorem 2, Proposition, and Theorem only for the vector fields having no closed orbit. The reason is that even if there is a homeomorphism $h$ on $M$ preserving the orbits of the vector fields $X$ and $Y$, we can find no relation between the period of a closed orbit $\gamma$ of $X$ and the period of the closed orbit $h(\gamma)$ of $Y$.

## References


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