17. *Cohomology of the Symmetric Space EI*

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§0. **Introduction.** Among the compact 1-connected irreducible symmetric spaces of exceptional type, $\boldsymbol{G}, \boldsymbol{FI}, \boldsymbol{EI}, \boldsymbol{EII}, \boldsymbol{EV}, \boldsymbol{EVI}, \boldsymbol{EVIII}$ and $\boldsymbol{EIX}$ have 2-torsion. The cohomology of $\boldsymbol{G}, \boldsymbol{FI}$ and $\boldsymbol{EII}$ are known [2], [4], [5]. In this paper we determine the cohomology of $\boldsymbol{EI}=E_6/PSp(4)$.

Since $\boldsymbol{EI}$ is 1-connected and $\pi_2(\boldsymbol{EI})=\boldsymbol{Z}_2$, we have two fiberings

(a) $\widetilde{\boldsymbol{EI}} \xrightarrow{g} \boldsymbol{EI} \xrightarrow{f} K(\boldsymbol{Z}_2; 2)$,

(b) $E_6 \longrightarrow \widetilde{\boldsymbol{EI}} \longrightarrow BSp(4)$.

On the other hand $\boldsymbol{EIV}=E_6/F_4$, and the subgroups F_4 and $PSp(4)$ of E_6 contain the subgroup $U=S^3\cdot Sp(3)$ in common. Noticing that $F_4/U=\boldsymbol{FI}$ and $PSp(4)/U=\boldsymbol{HP}^3$, we have two more:

(c) $\boldsymbol{HP}^3 \xrightarrow{i} E_6/U \xrightarrow{p} \boldsymbol{EI}$,

(d) $\boldsymbol{FI} \longrightarrow E_6/U \longrightarrow \boldsymbol{EIV}$.

We calculate the Serre spectral sequence associated to these fiberings.

Throughout the paper we use the following notations (A being a ring):

$A\{x_i\}=\oplus A\cdot x_i$ and $\Delta(x_i)=A\{m\,;\, m$ is a simple monomial in $x_i\}$.

Then our results are

Theorem 1. $H^*(\boldsymbol{EI}; \boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_i\,; i=2,3,5,9,11,13,15,17,21,23]/I$, *where* $Sq^i x_{i+1}=x_{2i+1}$ $(i=1,2,4,8)$, $Sq^j x_{j+7}=x_{2j+7}$ $(j=4,8)$, $Sq^8 x_{13}=x_{21}$ *and I is the ideal generated by the following elements* $(x_5'=x_5+x_2x_3)$: $x_2^3+x_3^2$, $x_2^2x_i$ $(i\neq 2,13,21)$, x_j^2 $(j\neq 2,3)$, $x_5'x_9x_{17}$; $x_3x_{13}+x_5'x_{11}$, $x_5'x_{13}$, $x_9x_{13}+x_5'x_{17}$, $x_{17}x_{13}$, $x_3x_{21}+x_9x_{15}$, $x_5'x_{21}+x_9x_{17}$, x_9x_{21}, $x_{17}x_{21}$; $x_{17}x_{11}+x_5'x_{23}$, $x_5'x_{15}+x_3x_{17}+x_9x_{11}$, $x_{17}x_{15}+x_9x_{23}$, $x_{17}x_{23}+x_3x_5'x_9x_{23}$; $x_{11}x_{13}$, $x_{11}x_{15}+x_3x_{23}$, $x_{11}x_{21}+x_9x_{23}$, $x_{13}x_{15}+x_5'x_{23}$, $x_{15}x_{21}$, x_kx_{23} $(k=11,13,15,21)$.

Theorem 2. (i) *As a ring* $H^*(\boldsymbol{EI})/\mathrm{Tors.}\, H^*(\boldsymbol{EI})$ *is generated by* $\{e_i, e_j'\,; i=8,9,17\,; j=16,17,25,34\}$ *and*

$$H^*\left(\boldsymbol{EI}; \boldsymbol{Z}\left[\frac{1}{2}\right]\right)=\boldsymbol{Z}\left[\frac{1}{2}\right][e_8]/(e_8^3)\otimes\Lambda(e_9, e_{17}),$$

in which $e_{16}'=\dfrac{1}{4}e_8^2$, $e_{17}'=\dfrac{1}{2}e_8e_9$, $e_{25}'=\dfrac{1}{2}e_8e_{17}$ *and* $e_{34}'=\dfrac{1}{4}e_8e_9e_{17}$.

(ii) *There exist torsion elements* $\chi\in H^3(\boldsymbol{EI})$ *of order* 2 *and* $\omega_i \in H^i(\boldsymbol{EI})$ $(i=5,11,15,23)$ *of order* 4, *and*

Tors. $H^*(\boldsymbol{EI})=\boldsymbol{Z}_2\{\chi, \chi\omega_5\}\otimes\{1, e_9, \omega_{11}, \omega_{15}, e_{17}, \omega_{23}, e_9\omega_{23}\}$
$+\boldsymbol{Z}_2\{\chi^2, \chi^2\omega_{15}, \chi^2\omega_{23}, \chi e_9\omega_{15}, \chi e_9 e_{17}\,;\, \omega_5 e_9, \omega_5\omega_{11}, \omega_5 e_{17}, e_9\omega_{15}, \omega_5 e_9\omega_{23}\}$
$+\boldsymbol{Z}_4\{\omega_5, \omega_{11}, \omega_{15}, \omega_5\omega_{15}, \omega_{23}, \omega_5\omega_{23}, e_9\omega_{23}, \omega_{15}\omega_{23}\}$.

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§ 1. $H^*(\widetilde{\boldsymbol{EI}}\,;\boldsymbol{Z}_2)$. Consider the Serre spectral sequence associated to the fibering (b). The cohomology of the base and the fiber are (respectively)

$H^*(BSp(4)\,;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[p_1, p_2, p_3, p_4]$, $\deg p_i=4i$, and

$H^*(E_6\,;\boldsymbol{Z}_2)=\Lambda(x_3, x_5, x_6, x_9, x_{15}, x_{17}, x_{23})$, $\deg x_i=i$

where $Sq^i x_{i+1}=x_{2i+1}$ $(i=2,4,8)$, $Sq^8 x_{15}=x_{23}$, $x_3^2=x_6$ and $x_j^2=0$ $(j\neq 3)$.

Lemma 1. *In this spectral sequence x_i are transgressive:*

$\tau x_3=p_1$, $\tau x_{15}=p_2^2$, $\tau x_{23}=p_3^2$ *and trivial for the rest.*

Proof. It is easy to see that they are transgressive and that $\tau x_3=p_1$, $\tau x_j=0$ $(j=5,6,9,17)$. τx_{15} and τx_{23} are obtained as follows.

Put $\tau x_{15}=ap_4+bp_2^2$ $(a, b\in Z_2)$. Applying first Sq^8 and then Sq^4, we have $\tau x_{23}=ap_2p_4+bp_3^2$ and $0=\tau Sq^4 x_{23}=ap_3p_4$. Suppose $a=b=1$, then in $E_{28}^{*,*}$ $p_3p_4=0$, which contradicts to $p_3p_4=p_2^2p_3\neq 0$.

Next suppose $(a, b)=(0,0)$ or $(1,0)$. Then the Poincaré series $P.S.(E_\infty^{*,*},t)$ has a pole of order >1. But $P.S.(E_\infty^{*,*},t)=P.S.(H^*(\widetilde{\boldsymbol{EI}}\,;\boldsymbol{Z}_2),t)$ cannot have such a pole since there is a fibering $K(\boldsymbol{Z}_2,1)\to\widetilde{\boldsymbol{EI}}\to\boldsymbol{EI}$ with $P.S.(H^*(\boldsymbol{Z}_2\,;1,\boldsymbol{Z}_2),t)=1/(1-t)$. Thus we have $a=0$ and $b=1$.

From the previous lemma

Corollary 1. $H^*(\widetilde{\boldsymbol{EI}}\,;\boldsymbol{Z}_2)=\Lambda(x_5, x_6, x_8, x_9, x_{12}, x_{17})\otimes\boldsymbol{Z}_2[x_{16}]$ *where* $Sq^1x_5=x_6$, $Sq^4x_5=x_9$, $Sq^8x_9=x_{17}$, $Sq^4x_8=x_{12}$, $Sq^8x_{12}=x_8x_{12}$.

§ 2. **Fibering (a).** According to Serre [7],

$H^*(\boldsymbol{Z}_2\,;2,\boldsymbol{Z}_2)=\boldsymbol{Z}_2[u_i\,;\,i=2^m+1,\ m\geqq 0]$, $\quad Sq^{i-1}u_i=u_{2i-1}$.

Consider the spectral sequence associated to (a). We calculate d_r for $r\leqq 23$.

Lemma 2. (i) x_i $(i\neq 16)$ *and* $x_{20}=x_8x_{12}$ *are transgressive:*

$\tau x_5=u_2^3+u_3^2$, $\tau x_{2i}=u_2^2u_{2i-3}$, $\tau x_{2j-1}=u_j^2$ $(i=3,4,6,10\,;\ j=5,9)$

(ii) $d_r x_{16}=0$ $(r<5)$ *and* $d_5 x_{16}=(u_5+\varepsilon u_2u_3)\otimes x_{12}$ $(\varepsilon\in Z_2)$.

Using these data $E_r^{*,*}$ is computed for $r\leqq 23$. We rewrite, for convenience, $u_3\otimes x_i$ with v_{i+3} $(i=8,12,20)$ and $u_2^2\otimes x_j$ with v_{j+4} $(j=9,17)$. They are permanent. Under these notations we see that up to degree 23 $E_\infty^{*,*}$ is generated as a ring by u_i $(i=2,3,5,9,17)$ and v_j $(j=11,13,15,21,23)$. (The structure is omitted here.) This implies that there exist indecomposables $x_j\in H^j(\boldsymbol{EI}\,;\boldsymbol{Z}_2)$ $(j=11,13,15,21,23)$, which together with f^*u_i $(i=2,3,5,9,17)$ generate $H^*(\boldsymbol{EI}\,;\boldsymbol{Z}_2)$ up to degree 23.

§ 3. $H^*(\boldsymbol{E}_6/\boldsymbol{S}^3\cdot\boldsymbol{Sp}(3)\,;A)$ **for** $A=\boldsymbol{Z}$ **and** $\boldsymbol{Z}_2$. Now consider the fibering (d). According to [1] and [5],

$H^*(\boldsymbol{EIV})=\Lambda(z_9, z_{17})$,

$H^*(FI)=Z[\chi, f_4, f_8, f_{12}]/(2\chi, f_4\chi, \chi^3, f_4^3-12f_4f_8+8f_{12}, f_4f_{12}-3f_8^2, f_8^3-f_{12}^2)$,
$H^*(FI; Z_2)=Z_2[y_2, y_3, y_8, y_{12}]/(y_2^3+y_3^2, y_2^2y_3, y_8^2+y_2^2y_{12}, y_{12}^2+y_2^2y_8y_{12})$,
where the mod 2 reduction of χ, f_4, f_8 and f_{12} are y_3, y_2^2, y_8 and y_{12} respectively. Note that the base space is 8-connected. It follows that j^* is surjective, and the spectral sequence collapses. Investigating the multiplicative structure we have

Proposition. *$H^*(E_6/S^3\cdot Sp(3); A)=H^*(FI; A)\otimes H^*(EIV; A)$, where $A=Z$ or Z_2 and in the latter case the isomorphism commutes with Sq^i.*

Concerning E_6/U and the map $i: HP^3\to E_6/U$, we need the following. For suitable choice of generater $s\in H^4(HP^3)$,

Lemma 3. *$i^*f_{4j}=4s^j$ $(j=1,3)$ and $i^*f_8=2s^2$.*

Corollary 2. *$H^*\left(EI; Z\left[\frac{1}{2}\right]\right)=Z\left[\frac{1}{2}\right][e_8]/(e_8^3)\otimes\Lambda(z_9, z_{17})$, where $p^*e_8=f_4^2-8f_8$, $p^*z_9=z_9$ and $p^*z_{17}=z_{17}$ mod decomposables.*

§4. Proof of Theorem 1. In the spectral sequence associated to (c)

$$E_2^{*,*}=H^*(EI; A)\otimes H^*(HP^3; A)=\sum_{0\leq i<4} H^*(EI; A)\otimes s^i.$$

In the sequel to the end we use the following abbreviations;

$$H^n=H^n(EI; A) \quad\text{and}\quad E_r^n=\sum_{p+q=n} E_r^{p,q}.$$

(The coefficient ring A will often be omitted.)

Lemma 4. (i) *There exist unique elements $x_i\in H^i$ $(i=11, 13)$ such that $p^*x_{11}=y_3y_8$, $Sq^1x_{11}=x_3x_9$ and $p^*x_{13}=y_2^2z_9$.*

(ii) *Define $x_{15}=Sq^4x_{11}$, $x_{23}=Sq^8x_{15}$ and $x_{21}=Sq^8x_{13}$. Then they are indecomposable, and hence generate H^* together with x_i $(i=2, 3, 5, 9, 17)$ up to degree 23.*

Proof. Since $\operatorname{Ker} p^*\cap H^5=Z_2\{x_5+x_2x_3\}$, we have $\tau s=x_5+x_2x_3$, and applying Sq^4, $\tau(s^2)=x_9$. Then it follows that $E_\infty^8=Z_2\{x_3x_5\}+Z_2\{x_2^2\}\otimes s$ and $E_\infty^9=Z_2\{x_5+x_2x_3\}\otimes s$. Since $p^*(x_3x_5)=y_2^4, y_8$ represents $x_2^2\otimes s\in E_\infty^{4,4}$ while y_3 does $x_3\otimes 1$. But $(x_3\otimes 1)(x_2^2\otimes s)=x_2^2x_3\otimes s=0$, which implies that y_3y_8 is of lower filtration degree. Consequently y_3y_8 is a p^*-image. Likewise we see that $y_2^2z_9$ is also a p^*-image. Then the rest of the assertion follows easily.

Next note that $\dim EI=42$ and that the bilinear form $H^n\times H^{42-n}\to H^{42}=Z_2$ induced by cup-product is non-degenerate. Using it we obtain the fundamental class $\in H^{42}$, and then additive bases for H^n for all n. More precisely

Lemma 5. *Let $N=\{3, 5, 9, 17\}$ and $N'=\{5, 9, 17\}$, then we have*

$$H^*(EI; Z_2)=Z_2[x_2, x_3, x_5, x_9, x_{17}]/(x_2^3+x_3^2, x_2^2x_i, x_j^2, (x_2x_3+x_5)x_9x_{17};\ i\in N,\ j\in N')$$
$$+(\Lambda(x_2, x_3, x_5)+\Lambda(x_2, x_3)\cdot x_9)\otimes x_{11}+\Lambda(x_2, x_3, x_9)\otimes x_{15}$$
$$+\Lambda(x_2, x_3, x_5, x_9)\otimes x_{23}+Z_2\{1, x_2, x_2^2\}\otimes Z_2\{x_{13}, x_{21}, x_{13}x_{21}\}.$$

By use of the Sq^i and the Adem relations we obtain all the relations necessary to rewrite monomials which cannot appear in the previous lemma (e.g. $x_i x_k$ ($i \in N$ and $k=13, 21$), $x_{11}x_{13}$, $x_{11}x_{15}$, etc.) and Theorem 1 is obtained.

§5. Proof of Theorem 2. Consider the spectral sequence associated to (c) with integral coefficients. It is readily seen that H^* ($* \leqq 7$) is generated by $\chi \in H^3$ of order 2 and $\omega_5 = \tau s \in H^5$ of order 4 and that $p^*\chi = \chi$, $p^*\omega_5 = 0$ and $\chi\omega_5 \neq 0$. By Lemma 3, $f_4^2 - 8f_8 \in \operatorname{Ker} i^*$ and it represents an element in $E_\infty^{8-q,q}$ for some $q<8$, which is trivial if $q>0$. Consequently $f_4^2 - 8f_8 = p^*e_8$ for some $e_8 \in H^8$. We choose e_8 such that $e_8 \equiv x_2^4 \pmod 2$.

Next by Corollary 2 and Theorem 1, $H^9 = \boldsymbol{Z}\{e_9\}$ and $E_2^9 = \boldsymbol{Z}\{e_9\} + \boldsymbol{Z}_4\{\omega_5\} \otimes s$. On the other hand $H^9(E_6/U) = \boldsymbol{Z}\{z_9\}$ and $z_9 \bmod 2$ is not a p^*-image. Note that $d_5(s^2) = 2\omega_5 \otimes s$, and we have $E_\infty^9 = \boldsymbol{Z}\{e_9\} + \boldsymbol{Z}_2\{\omega_5\} \otimes s$, whence $p^*e_9 = \pm 2z_9$. Looking into the filtration on $H^*(E_6/U)$, e_{17} is similarly defined, and $p^*e_{17} \equiv 2z_{17}$ mod decomposables.

Since $e_8^2 \equiv x_2^8 = 0 \bmod 2$, e_8^2 is divisible by 2. Calculation in $H^*(E_6/U)$ shows that there is an element e'_{16} such that $e_8^2 = 4e'_{16}$, and $H^{16} = \boldsymbol{Z}\{e'_{16}\}$. For the similar reason we must introduce three more: $e'_{17} = \frac{1}{2}e_8 e_9$, $e'_{25} = \frac{1}{2}e_8 e_{17}$ and $e'_{34} = \frac{1}{4}e_8 e_9 e_{17}$.

At last for Tors. $H^*(\boldsymbol{EI})$. In E_r^* we see that $\chi^2 \otimes s$ is not a cocycle since $H^{10} = 0$, from which $\chi^2\omega_5 = d_5(\chi^2 \otimes s) \neq 0$. But $\chi^2\omega_5 \equiv x_3^2 x_5 \bmod 2$, which vanishes. So we have another higher torsion ω_{11} such that $2\omega_{11} = \chi^2\omega_5$. By similar argument we have two more: ω_{15} and ω_{23}. It is seen that for $i = 5, 11, 15$ and 23

$$\omega_i \bmod 2 = x_i + x_2 x_{i-2} \equiv \beta' x_2 x_3 x_{i-6} \bmod \operatorname{Im} \beta \qquad (i \neq 5),$$

where $\beta = \frac{1}{2}\delta$ and $\beta' = \frac{1}{4}\delta$ are Bockstein operations. Using the derivativity of β and β' and comparing with the mod 2 cohomology, we obtain the structure of Tors. $H^*(\boldsymbol{EI})$. From the procedure Theorem 2 follows.

References

[1] S. Araki: Cohomology modulo 2 of the compact exceptional groups E_6 and E_7. J. Math. Osaka City Univ., **12**, 43–65 (1961).

[2] A. Borel and F. Hirzebruch: Characteristic classes and homogeneous spaces. Amer. J. Math., **80**, 458–538 (1958).

[3] E. Cartan: Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple. Ann. Sci. Ecole Norm. Sup., **44**, 345–467 (1927).

[4] K. Ishitoya: Integral cohomology ring of the symmetric space ***EII*** (to

appear in J. Math. Kyoto Univ.).
[5] K. Ishitoya and H. Toda: On the cohomology of irreducible symmetric spaces of exceptional type (to appear).
[6] J.-P. Serre: Homologie singulière des espaces fibrés. Ann. of Math., **54**, 425–505 (1951).
[7] ——: Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. Comment. Math. Helv., **27**, 198–232 (1953).