13. *Bifurcation of Stable Stationary Solutions from Symmetric Modes*

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Introduction. We consider the following semilinear parabolic system of equations:

$$\begin{aligned} &U_t = D(\sigma)U_{xx} + BU + F(U), \qquad (t,x) \in (0,+\infty)\times(0,L) \\ &U(t,0) = U(t,L) = 0, \end{aligned} \tag{P-1}$$

where $U = {}^t(u(t,x), v(t,x))$, $D(\sigma) = (D_u(\sigma), D_v(\sigma))$ and σ is a real parameter, $B = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ is a real constant matrix and $F(U) = {}^t(f_1(u,v), f_2(u,v))$ is a smooth autonomous nonlinear operator which satisfies

$$F(0) = F_U(0) = 0. \tag{0-1}$$

We assume that B satisfies either of the following conditions:

$$\det B > 0, \ a > 0, \qquad d = 0, \tag{0-2}$$

$$\det B > 0, \ a > 0, \qquad a + d \leqq 0. \tag{0-3}$$

Our main purpose is to show the existence of bifurcation of stable stationary solutions of (P-1) as $D(\sigma)$ varies. Stationary problem of (P-1) and its linearized system of equations at $U=0$ are given as follows:

$$\begin{aligned} &D(\sigma)U_{xx} + BU + F(U) = 0, \\ &U(0) = U(L) = 0, \end{aligned} \tag{P-2}$$

$$\begin{aligned} &D(\sigma)U_{xx} + BU = 0, \\ &U(0) = U(L) = 0. \end{aligned} \tag{P-3}$$

Section 1 deals with the spectrum of (P-3) and the existence of bifurcation of stationary solutions from any mode of the eigenfunction of (P-3) under the appropriate conditions of $D(\sigma)$ and B. Section 2 deals with the asymptotic stability of the bifurcating solutions from symmetric modes. In section 3 we give some examples of biological system to which our theorems can apply.

§ 1. Existence. Using the Fourier series expansion of U,

$$U = \sum_{n=1}^{\infty} U_n \sin \frac{n\pi}{L} x = \sum_{n=1}^{\infty} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \sin \frac{n\pi}{L} x,$$

we obtain the infinite system of linear equations of $\{U_n\}_{n\in N}$:

$$M_n U_n = 0, \quad M_n = \begin{pmatrix} -D_u\left(\dfrac{\pi}{L}\right)^2 n^2 + a, & b \\ c, & -D_v\left(\dfrac{\pi}{L}\right)^2 n^2 + d \end{pmatrix}, \quad n \in \boldsymbol{N}.$$

The roots $\{\alpha_n^i\}_{i=1,2}$ $(\operatorname{Re}\alpha_n^1 \geqq \operatorname{Re}\alpha_n^2)$ of the characteristic equation

$\det(M_n-\alpha I)=0$ are the eigenvalues of (P-3) which correspond to the $\sin n\pi/L$-mode. We consider the following condition of the spectrum of (P-3):

$\alpha_{n_0}^1=0$, $\mathrm{Re}\,\alpha_n^i<0$ for all $(i,n)\in\{1,2\}\times N$ except $(i,n)=(1,n_0)$. (S_{n_0})
The corresponding eigenfunction to $\alpha_{n_0}^1$ is denoted by $U_{n_0}\sin(n_0\pi/L)x$. The necessary and sufficient conditions of $D(\sigma)$ and B to realize the condition (S_{n_0}) are given in the following lemma.

For simplicity we write D instead of $D(\sigma)$. We introduce the following curves in $D^+=\{(D_u,D_v)\,;\,D_u>0,\ D_v>0\}$-plane:

$$H_n: D_v=\frac{bc}{(\gamma n^2)^2}\cdot\frac{1}{D_u-a/\gamma n^2}+\frac{d}{\gamma n^2},\quad \gamma=\left(\frac{\pi}{L}\right)^2,\quad n\in N,$$

$$L: D_u+D_v=\frac{a}{\gamma},$$

$P^n=(P_u^n,P_v^n)$ is a cross point of H_n and H_{n+1} and

$L^n=(L_u^n,L_v^n)$ is a cross point of L and H_n.

Note that P_u^n and P_v^n are strictly decreasing with respect to n.

Lemma 1. B$_1$) *Suppose that B satisfies* (0-2). *Then S_1 holds if and only if $D\in H_1$ and $D_u>P_u^1$, and for $n_0\geqq 2$, S_{n_0} holds if and only if $D\in H_{n_0}$,* $\max\{P_u^{n_0},L_u^{n_0}\}<D_u<P_u^{n_0-1}$ *and*

$$-\frac{bc}{a^2}>I(n_0)=\frac{2n_0^3(n_0-1)^3}{\{n_0^2+(n_0-1)^2\}^2}. \tag{1-1}$$

B$_2$) *Suppose that B satisfies* (0-3). *Then for each $n_0\in N$, S_{n_0} holds if and only if $D\in H_{n_0}$ and $P_u^{n_0}<D_u<P_u^{n_0-1}$ ($P_u^0=+\infty$ for convention).*

In the following we consider the bifurcation problem of (P-2) as $D(\sigma)$ crosses the bifurcation curve stated in Lemma 1. We assume that $D(\sigma)$ satisfies the following two conditions:

1) $D(\sigma)$ is a smooth vector-valued function of σ defined in the neighborhood of $\sigma=0$ and $D_0=D(0)$ is on the bifurcation curve in Lemma 1, i.e., there exists an $n_0\in N$ and $D_0\in H_{n_0}$. (1-2)

2) $(d/d\sigma)D(\sigma)|_{\sigma=0}=D'(0)\neq 0$ and the vector $D'(0)$ intersects transversally with the curve H_{n_0} at D_0. (1-3)

Using the Theorem 2.4 of [1], we obtain the next theorem.

Theorem 1. *Suppose that* (0-1), (0-2) (*or* (0-3)), (1-2) *and* (1-3) *hold and that in case* (B$_1$) *B satisfies the inequality* (1-1) *besides* (0-2). *Then there exists a unique one-parameter family of nontrivial classical solutions $(D(\sigma(s)),U(s))$ of* (P-2) *for $|s|<{}^\exists s_0$ such that $\sigma(s)$ and $U(s)$ are smooth with respect to s and*

$$U(s)=sU_{n_0}\sin\frac{n_0\pi}{L}x+o(s)\qquad \text{as } s\to 0$$

and

$$\sigma(0)=0.$$

§2. **Nonlinear stability.** For simplicity we assume that $F(U)$ is real analytic in this section, i.e., $f_i(u, v)$ is a real analytic function with respect to u and v, $i=1, 2$.

The linearized stability of the bifurcating solution $U(s)$ is determined by the bifurcation direction, i.e., the form of $\sigma(s)$ near $s=0$ (cf. [2]). In Lemma 2 we give a simple criterion of the bifurcation direction when n_0 is an odd number. (Note that $U_{n_0} \sin (n_0\pi/L)x$ is symmetric with respect to x when n_0 is odd.)

Using the methods of [3] and [4], we can prove the nonlinear stability or instability of $U(s)$ bifurcating from symmetric modes.

Lemma 2. *Suppose that the assumptions of Theorem* 1 *hold and let $Q(U)$ be a quadratic part of $F(U)$ and let $U^*_{n_0} \sin (n_0\pi/L)x$ be an eigenfunction of the adjoint equation of* (P-3) *which corresponds to the zero eigenvalue. Then if n_0 is odd, $\dot{\sigma}(0)\neq 0$ $(\cdot = d/ds)$ if and only if*

$$\int_0^L \left(Q\left(U_{n_0} \sin \frac{n_0\pi}{L}x\right), U^*_{n_0} \sin \frac{n_0\pi}{L}x\right)dx \neq 0. \tag{C}$$

Here (,) *denotes the usual inner product in R^2.*

Remark 1. If n_0 is even, the bifurcating solution $U(s)$ in Theorem 1 is in general unstable. We shall study about this in a forthcoming paper.

We note that the criterion (C) in Lemma 2 is a fairly general condition and is satisfied by almost all the nonlinear operators.

From the relation between bifurcation direction and a critical eigenvalue in Theorem 1.16 of [2], we obtain the following lemma about linearized stability.

Lemma 3. *Let n_0 be odd and assume that the criterion* (C) *holds. Then the bifurcation occurs on both sides of the bifurcation curve H_{n_0}, i.e., $D(\sigma(s))$ intersects transversally with H_{n_0} as s moves in $(-s_0, s_0)$. (Therefore the curve $D(\sigma(s))$, $|s|<s_0$ is divided into two parts, i.e., one is on the upper side of H_{n_0} and another is on the lower side of it.) And the upper side bifurcating solutions are stable and the lower side bifurcating ones are unstable in a linearized sense.*

The perturbed system of equations from $U(s)$ is obtained by inserting $U=U(s)+W$ into (P-1) as follows:

$$\begin{aligned} &W_t = D(\sigma(s))W_{xx} + BW + F_U(U(s))W + G(W; U(s)),\\ &W(t, 0) = W(t, L) = 0,\\ &W(0, x) = W_0, \end{aligned} \tag{P-4}$$

where

$$G(W; U(s)) = F(U(s)+W) - F(U(s)) - F_U(U(s))W.$$

Let us define the following two linear operators in $E=(L^2(0, L))^2$ with norm $\|\cdot\|$:

$$A = -D(\sigma(s))\frac{\partial^2}{\partial x^2}, \qquad D(A) = (H^2(0, L))^2 \cap (H^1_0(0, L))^2.$$

$$\tilde{A}=A-B-F_U(U(s)), \qquad D(\tilde{A})=D(A).$$

Using the results of [4], we conclude from Lemma 3:

Theorem 2. *Let the assumptions of Theorem* 1 *and Lemma* 3 *hold. Then the upper side bifurcating solutions $U(s)$ are asymptotically stable in the topology of $D(A^\alpha)$ $(1/2\leqq\alpha<1)$, i.e., for any $\varepsilon>0$ there exists a positive number $\delta(\varepsilon)$ and if $\|A^\alpha W_0\|<\delta(\varepsilon)$,* (P-4) *has a global strict solution and we have*

$$\|A^\alpha W(t)\|\leqq\varepsilon e^{-bt}, \qquad t\in[0,+\infty).$$

The value $b>0$ is determined by the spectrum of $\tilde{A}$, i.e., $0<b<\mathrm{Re}\,(\tilde{A})$.

As for the lower side bifurcating solutions, they are unstable in the topology of E.

§ 3. Examples. 1) We consider the following system of equations (cf. [5]):

$$\begin{aligned} &u_t=D_u u_{xx}+(2+u-u^2)u-uv,\\ &v_t=D_v v_{xx}-gv+uv, \qquad\qquad (3\text{-}1)\\ &u(t,0)=u(t,L)=u_0, \qquad v(t,0)=v(t,L)=v_0, \end{aligned}$$

where g is a constant such that $0<g<1/2$ and (u_0, v_0) is a unique positive constant solution of (3-1).

Applying the following transformation to (3-1),

$$\hat{u}=u-u_0, \qquad \hat{v}=v-v_0, \qquad (3\text{-}2)$$

we obtain the system of equations:

$$\begin{aligned} &\hat{u}_t=D_u\hat{u}_{xx}+(1-2g)g\hat{u}-g\hat{v}-\hat{u}^3+(1-3g)\hat{u}^2-\hat{u}\hat{v},\\ &\hat{v}_t=D_v\hat{v}_{xx}+v_0\hat{u}+\hat{u}\hat{v}, \qquad\qquad (3\text{-}3)\\ &\hat{u}(t,0)=\hat{u}(t,L)=0, \qquad \hat{v}(t,0)=\hat{v}(t,L)=0. \end{aligned}$$

It is easy to see that this system corresponds to the case (0-2), and we can apply Theorems 1 and 2 to (3-3).

2) (M. Mimura's patchiness model.) Next we consider the following system of equations:

$$\begin{aligned} &u_t=D_u u_{xx}+\left(\frac{1}{9}(-u^2+16u+35)-v\right)u,\\ &v_t=D_v v_{xx}+\left(-\left(1+\frac{2}{5}v\right)+u\right)v, \qquad\qquad (3\text{-}4)\\ &u(t,0)=u(t,L)=5, \qquad v(t,0)=v(t,L)=10, \end{aligned}$$

where (5, 10) is a unique positive constant solution of (3-4). Applying the same procedure to (3-4) as in 1), we get a system of equations which corresponds to the case (0-3) and to which we can apply Theorems 1 and 2.

References

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