11. *On the Acyclicity of Free Cobar Constructions. I*

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(Communicated by Kunihiko KODAIRA, M. J. A., April 12, 1977)

1. Let Λ be a group ring or an enveloping algebra of Lie algebras over $\boldsymbol{Z}$ with an augmentation $\varepsilon: \Lambda \to \boldsymbol{Z} \to (0)$. Let (X, ∂) be a complex of left Λ-modules

$$(1.1)\qquad \longrightarrow X_p \xrightarrow{\partial_p} X_{p-1} \xrightarrow{\partial_{p-1}} \cdots \longrightarrow X_1 \xrightarrow{\partial_1} \Lambda \xrightarrow{\varepsilon} \boldsymbol{Z} \longrightarrow (0)$$

where each X_p is a free left Λ-module and each ∂_p is a left Λ-module homomorphism. Let Λ^f be a free associative algebra over $\boldsymbol{Z}$ such that we get an exact sequence

$$(1.2)\qquad (0) \longrightarrow L \xrightarrow{\iota_0} \Lambda^f \xrightarrow{\kappa_0} \Lambda \longrightarrow (0)$$

where L denotes an ideal of Λ^f. First we assume

Assumption 1. i) *There exist two sequences* (X^f, ∂^f) *and* $(L \otimes_{\Lambda^f} X^f, 1 \otimes \partial^f)$ *of left* Λ^f*-modules on the augmentation* ε^f,

$$(1.3)\qquad (0) \longrightarrow (\Lambda^f)^+ \longrightarrow \Lambda^f \xrightarrow{\varepsilon^f} \boldsymbol{Z} \longrightarrow (0)$$

such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
 & (0) & & (0) & & & & (0) & & (0) & \\
 & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
\longrightarrow & X_p & \xrightarrow{\partial_p} & X_{p-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \xrightarrow{\partial_1} & \Lambda & \longrightarrow \boldsymbol{Z} \longrightarrow (0) \\
 & \uparrow{\scriptstyle \kappa_p} & & \uparrow{\scriptstyle \kappa_{p-1}} & & & & \uparrow{\scriptstyle \kappa_1} & & \uparrow{\scriptstyle \kappa_0} & \\
\longrightarrow & X^f_p & \xrightarrow{\partial^f_p} & X^f_{p-1} & \longrightarrow & \cdots & \longrightarrow & X^f_1 & \xrightarrow{\partial^f_1} & \Lambda^f & \longrightarrow \boldsymbol{Z} \longrightarrow (0) \\
 & \uparrow{\scriptstyle \iota_p} & & \uparrow{\scriptstyle \iota_{p-1}} & & & & \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_0} & \\
\longrightarrow & L \otimes_{\Lambda^f} X^f_p & \xrightarrow{1\otimes\partial^f_p} & L \otimes_{\Lambda^f} X^f_{p-1} & \longrightarrow & \cdots & \longrightarrow & L \otimes_{\Lambda^f} X^f_1 & \xrightarrow{1\otimes\partial^f_1} & L & \\
 & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
 & (0) & & (0) & & & & (0) & & (0) &
\end{array}$$

where each κ_p *denotes an epimorphism as left modules and each* X_p *or* X^f_p *is isomorphic to the tensor product* $\Lambda \otimes S_p$ *or* $\Lambda^f \otimes S_p$ *respectively for a free abelian group* S_p *such that* $L = X^f_2 \otimes \Lambda^f$.

ii) ∂^f_p *are all injective on* X^f_p, $p \geqq 1$.

Let $\overset{+}{X}{}^f$ be the direct sum $\bigoplus_{p=2}^{\infty} X^f_p \oplus (\Lambda^f)^+$ and $T(\overset{+}{X}{}^f)$ be the tensor algebra of $\overset{+}{X}{}^f$ denoted by A. A becomes a free graded algebra $\bigoplus_{s=0}^{\infty} A_s$, where A_s is spanned by the elements of the form $u_1 u_2 \cdots u_{m-1} u_m$, $u_j \in X^f_{p_j}$ $(p_j \geqq 2)$, $j \leqq m-1$, and $u_m \in \Lambda^f$ such that $s = \sum_{j=1}^{m-1} (p_j - 1)$.

We introduce a lexicographic order into a basis of A : Let $\{u^p_\gamma, \gamma \in \Gamma_p\}$ be an ordered basis of S_{p+1}, $p \geqq 1$ and $\{u^0_\gamma, \gamma \in \Gamma_0\}$ be an ordered system of generators of $(\Lambda^f)^+$. Then the elements of the form $u_{\gamma_1}^{p_1}\cdot u_{\gamma_2}^{p_2}\cdots u_{\gamma_m}^{p_m}$, $\gamma_\nu\in\Gamma_{p_\nu}$, $p_\nu\geqq 0$, form a basis of A.

Definition. We say that u_α^p, $\alpha\in\Gamma_p$ is greater than u_β^q, $\beta\in\Gamma_q$, if $p>q$ or $p=q$ and $\alpha>\beta$, and that an element $u_{\alpha_1}^{p_1}\cdots u_{\alpha_m}^{p_m}$ is greater than an element $u_{\beta_1}^{q_1}\cdots u_{\beta_m}^{q_m}$ if $p_m=q_m$, $\alpha_m=\beta_m$, $\cdots$, $p_{m-k+1}=q_{n-k+1}$, $\alpha_{m-k+1}=\beta_{n-k+1}$ and $p_{m-k}>q_{n-k}$, or $p_m=q_n$, $\alpha_m=\beta_n$, $\cdots$, $p_{m-k+1}=q_{n-k+1}$, $\alpha_{m-k+1}=\beta_{n-k+1}$, $p_{m-k}=q_{n-k}$ and $\alpha_{m-k}>\beta_{n-k}$ for a certain k. Let $A(p_1, p_2,\cdots,p_m)$ be the left Λ^f-submodule of A generated by the elements $u_{\gamma_1}^{p_1}\cdot u_{\gamma_2}^{p_2}\cdots u_{\gamma_m}^{p_m}$, $\gamma_\nu\in\Gamma_{p_\nu}$, $p_1\geqq 1$ and by $\mathcal{A}(p_1,\cdots,p_m)$ be the direct sum of all the $A(q_1,\cdots q_n)$ such that $(q_1,\cdots,q_n)<(p_1,\cdots,p_m)$, An element of $A(p_1,\cdots,p_m)$ will be called of type $(p_1,\cdots,p_m)$, and the type $(p_1,\cdots,p_m)$ will be said to be higher than the type $(q_1,\cdots,q_n)$ if $p_m=q_n,\cdots,p_{m-k+1}=q_{n-k+1}$ and $p_{m-k}>q_{n-k}$ for a certain k. We next assume

Assumption 2. *There exist boundary operators* $\delta=\{\delta_s\}$, $\delta_s: A_s\to A_{s-1}$ *such that*

i) $\delta_s u^s\equiv\partial_{s+1}u^s \bmod \mathcal{A}(s-1)$, $u^s\in S_{s+1}$,

ii) $\delta u_1u_2\cdots u_{m-1}u_m=\sum_{\nu=1}^m(-1)^{\sum_{j=1}^{\nu-1}\deg u_j}u_1u_2\cdots u_{\nu-1}\cdot\delta u_\nu\cdots u_{m-1}\cdot u_m$,

iii) $\delta_1(A_1)$ *coincides with* L.

Then we have the complex (A,δ) such that δ preserves each $\mathcal{A}(p_1,\cdots,p_m)$. This complex will be called the "*free cobar construction*" of the complex (X,∂). By induction procedure with respect to the types we can prove the

Main Theorem. *Under Assumptions* 1 *and* 2 *we have*

$$\mathrm{H}_p(A)\cong(0),\ p\geqq 1 \quad\text{and}\quad \mathrm{H}_0(A)\cong\Lambda,\ p=0, \tag{1.5}$$

if and only if (1.1) is a free resolution of Λ.

References

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