9. *Some Remarks on Kodaira Dimentions of Fiber Spaces**)

By Takao FUJITA
College of General Education, University of Tokyo

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This note is a development of recent results of Ueno [5]. Details will be published elsewhere. The author would like to express his hearty thanks to Professor Iitaka, Professor Ueno and Professor Popp who supported him during the preparation of this note.

Every manifold here is assumed to be projective. Sometimes we denote the pull-back of a given line bundle L by L by abuse of notation. Otherwise the notation is essentially the same as in [3], [4] and [5].

Proposition 1. *Let $\pi: M \to S$ be a fiber space and let L and H be line bundles on M and S respectively. Suppose that $\kappa(S,H)=\dim S$ and that $\kappa(M, aL-bH_M)\geqq 0$ for certain positive integers a, b. Then $\kappa(M,L)=\kappa(F,L_F)+\kappa(S,H)$ for a general fiber F of π.*

Proof. The Iitaka-inequality ([3] Th. 4 or [4] Th. 5.11) says that $\kappa(M,L)\leqq\kappa(F,L)+\dim S$. So it suffices to show $\kappa(M,L)\geqq\kappa(F,L)+\kappa(S,H)$. Now, taking a sufficiently large multiple of L instead of L, we may assume that the rational mapping $\rho_{|L|}: M\to V\subset \boldsymbol{P}^{\dim|L|}$ gives an Iitaka-fibering of M with respect to L ([3] or [4] Th. 5.10). Similarly we may assume that $\rho_{|H|}: S\to W\subset\boldsymbol{P}^{\dim|H|}$ is birational onto the image W. Moreover we may assume that $\Gamma(M,L-H)\neq 0$. An element $\alpha\neq 0$ of $\Gamma(M,L-H)$ defines an injection $\Gamma(S,H)\to\Gamma(M,L)$, so a projection $\boldsymbol{P}^{\dim|L|}\to\boldsymbol{P}^{\dim|H|}$, and consequently a rational mapping $V\to W$. Making everything smooth and holomorphic by birational modifications, we obtain the following diagram:

$$\begin{array}{ccc} M^* & \xrightarrow{f} & V^* \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle h} \\ S^* & \xrightarrow{g} & W^*. \end{array}$$

Take a general point w on W and observe the mapping $f_w: M^*_w=(g\circ\pi^*)^{-1}(w)\to V^*_w=h^{-1}(w)$. A general fiber G of f_w is also general as a fiber of f, hence $\kappa(G,L)=0$ since f is an Iitaka-fibering with respect to L. So Iitaka's inequality implies $\kappa(M^*_w,L)\leqq\kappa(G,L)+\dim V^*_w=\dim V^*-\dim W^*=\kappa(M,L)-\kappa(S,H)$. On the other hand, M^*_w is birational to a general fiber F of π, hence $\kappa(M^*_w,L)=\kappa(F,L)$. Thus we obtain the desired inequality.

As an example of applications of this criterion, we give an outline of our proof of the following

Theorem. *Let $\pi: M \to C$ be a fiber space over a curve C of genus $g \geqq 2$. Suppose that $\kappa(M) \geqq 0$ or $p_g(F) > 0$ for a general fiber F of π. Then $\kappa(M) = 1 + \kappa(F)$.*

We need several lemmata.

Lemma 1. *Let M be a manifold with $\kappa(M) \geqq 0$. Then there is a surjective morphism $f: N \to M$ from a manifold N with* $\dim N = \dim M$, *$\kappa(N) = \kappa(M)$ and $p_g(N) > 0$.*

Outline of the proof. For a member $D \in |kK_M|$ we construct in a natural way a subvariety W in K_M such that the projection $K_M \to M$ restricted to W makes W a cyclic k-sheeted branched covering of M with branch locus D. A smooth model N of W has the desired property.

Lemma 2. *Let $\pi: M \to C$ be a fiber space over a curve C. Then $\pi_* \omega_M$ $(\omega_M = \mathcal{O}_M[K_M])$ is a direct sum of $\omega_C \oplus \cdots \oplus \omega_C$ ($h^1(C, \pi_* \omega_M)$-times) and a locally free sheaf $\mathcal{E}$ with $h^1(C, \mathcal{E}) = 0$.*

Our proof is based on the analysis of the Leray spectral sequence of ω_M with respect to π and on the theory of Hodge decomposition.

Lemma 3. *Let E be a vector bundle over a curve C with $g(C) \geqq 2$ and suppose that $h^1(C, E) = 0$. Then E is ample in the sense of Hartshorne.*

We can prove this lemma by induction on rank E.

Now we prove Theorem. In view of Lemma 1, we may assume $p_g(F) > 0$. So $\pi_* \omega_M \neq 0$. Combining Lemma 2 and Lemma 3 we infer that $\pi_* \omega_M$ is ample. Hence $S^k(\pi_* \omega_M)[-K_C]$ is generated by global sections for $k \gg 0$. Consequently $\Gamma(C, \pi_*(\omega_M^{\otimes k})[-K_C]) \neq 0$ since there is a natural homomorphism $S^k(\pi_* \omega_M) \to \pi_*(\omega_M^{\otimes k})$. Therefore $\Gamma(M, kK_M - K_C) \neq 0$ and we apply Proposition 1 to prove the theorem.

Together with the results of Viehweg [6] and Ueno [5], Theorem implies the following

Corollary. *Let M be a threefold with $\kappa(M) = 0$. Then the Albanese mapping $M \to$* Alb *(M) is surjective.*

As another application of Proposition 1, we give the following

Proposition 2. *Let $\pi: M \to S$ be a fiber space over a manifold S with $\kappa(S) = \dim S$. Suppose that $\pi_* \omega_{M/S} (\omega_{M/S} = \mathcal{O}_M[K_M - K_S])$ is locally free and numerically semipositive. Then $\kappa(M) = \kappa(F) + \kappa(S)$ for a general fiber F of π.*

Proof. Let H be an ample line bundle on S. We have $\kappa(S, kK_S - H) \geqq 0$ for some $k > 0$ since $\kappa(S) = \dim S$. From the semipositivity we infer that $\kappa(M, H + t(K_M - K_S)) \geqq 0$ for any $t > 0$. Take t so large that $t > k$. Then $\kappa(M, tK_M - K_S) \geqq \kappa(M, kK_S + t(K_M - K_S)) \geqq \kappa(M, H + t(K_M - K_S)) \geqq 0$. Now we can apply Proposition 1.

Remark 1. $\pi_* \omega_{M/S}$ is locally free and semi-positive if π is everywhere of maximal rank ([1]), or if dim $S=1=p_g(F)$ (see a forthcoming paper of the author).

Remark 2. If we can prove the addition inequality of Kodaira dimensions for any fiber space over a manifold of hyperbolic type, then follows the surjectivity of the Albanese mapping for any manifold of parabolic type (see [4] & [5]).

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