# 62. The Fourier Transform of the Schwartz Space on a Symmetric Space

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(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1977)

**1. Introduction.** Let $S$ be a symmetric space of the noncompact type and let $L^2(S)$ denote the space of square integrable functions on $S$ with respect to the invariant measure. In his paper [7], S. Helgason characterized the image of $L^2(S)$ by the Fourier transform.

The purpose of this paper is to give a characterization of the image of the Harish-Chandra's Schwartz space by the Fourier transform. As an immediate consequence we obtain the above mentioned result of S. Helgason (the characterization of the image of $L^2(S)$ by the Fourier transform). The proofs of the results are given in [2].

**2. Notation and preliminaries.** If $M$ is a manifold (satisfying the second countability axiom), following Schwartz $\mathcal{D}(M)$ denotes the the space of $C^\infty$ functions on $M$ with compact support. If $V$ is a real vector space $\mathcal{S}(V)$ denotes the space of rapidly decreasing functions on $V$ (see [8]) and $\boldsymbol{D}(V)$ denotes the algebra of differential operators with constant coefficients on $V$.

If $G$ is a Lie group and $H$ a closed subgroup, $G/H$ denotes the space of left cosets $gH$, $g \in G$. $\boldsymbol{D}(G/H)$ denotes the algebra of differential operators on homogeneous space $G/H$ which are invariant under left translations by $G$. We write $\boldsymbol{D}(G)$ for $\boldsymbol{D}(G/e)$, where $e$ is the identity of $G$.

Let $S$ be a symmetric space of the noncompact type that is a coset space $S=G/K$ where $G$ is a connected semisimple Lie group with finite center and $K$ a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G$ and $K$ respectively. Let $B$ be the Killing form of $\mathfrak{g}$ and $\theta$ the Cartan involution which associates with the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace and $\mathfrak{a}^*$ its dual. Put $A=\exp\mathfrak{a}$. For $\lambda\in\mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda=\{X\in\mathfrak{g}\mid [H,X]=\lambda(H)X, \text{ for all } H\in\mathfrak{a}\}.$$

If $\lambda\neq 0$ and $\mathfrak{g}_\lambda\neq\{0\}$ then $\lambda$ is called a restricted root and $m_\lambda=\dim(\mathfrak{g}_\lambda)$ is called its multiplicity. Let $\mathfrak{g}_c$ and $\mathfrak{a}_c^*$ denote the complexifications of $\mathfrak{g}$ and $\mathfrak{a}^*$ respectively. If $\lambda$, $\mu\in\mathfrak{a}_c^*$ let $H_\lambda\in\mathfrak{a}_c$ (the complex subspace of $\mathfrak{g}_c$ spanned by $\mathfrak{a}$) be determined by $\lambda(H)=\langle H_\lambda, H\rangle$ for $H\in\mathfrak{a}$ and put $\langle\lambda,\mu\rangle=\langle H_\lambda, H_\mu\rangle$. Since $B$ is positive definite on $\mathfrak{p}$ we put $\|\lambda\|=\langle\lambda,\lambda\rangle^{1/2}$

for $\lambda \in \mathfrak{a}^*$ and $\|X\| = B(X, X)^{1/2}$ for $X \in \mathfrak{p}$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}'$ and call a restricted root $\alpha$ positive if it is positive on $\mathfrak{a}^+$. Let $\mathfrak{a}^*_+$ denote the corresponding Weyl chamber in $\mathfrak{a}^*$. Let $\sum$ denote the set of restricted roots, $P_+$ the set of positive roots. Put $\rho = (1/2)\sum_{\alpha \in P_+} \alpha$, $\mathfrak{n} = \sum_{\alpha \in P_+} \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$. Let $N$ and $\bar{N}$ denote the corresponding analytic subgroups of $G$. Thus we obtain an Iwasawa decomposition $G = KAN$ of $G$. For each $g \in G$ can be uniquely written $g = \kappa(g) \exp H(g) n(g)$, $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Let $M$ denote the centralizer of $A$ in $K$, $M'$ the normalizer of $A$ in $K$, $W$ the factor group $M'/M$, the Weyl group. The group $W$ acts as a group of linear transformations of $\mathfrak{a}^*$ by $(s\lambda)(H) = \lambda(s^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*$ and $s \in W$. Let $w$ denote the order of $W$.

Let $l = \dim \mathfrak{a}$. The Killing form induces Euclidean measures on $A$, $\mathfrak{a}$ and $\mathfrak{a}^*$; multiplying these by the factor $(2\pi)^{-(1/2)l}$ we obtain invariant measures $da$, $dH$ and $d\lambda$, and the inversion formula for the Fourier transform

$$f^*(\lambda) = \int_A f(a) e^{-i\lambda(\log a)} da, \qquad \lambda \in \mathfrak{a}^*, \tag{1}$$

$$f(a) = \int_{\mathfrak{a}^*} f^*(\lambda) e^{i\lambda(\log a)} d\lambda, \qquad f \in \mathcal{S}(A) \tag{2}$$

holds without multiplicative constant.

We normalize the Haar measures $dk$ and $dm$ on the compact groups $K$ and $M$, respectively, so that the total measure is 1. The Haar measures of the nilpotent groups $N$, $\bar{N}$ are normalized so that

$$\theta(dn) = d\bar{n}, \qquad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1.$$

The Haar measure $dg$ on $G$ can be normalized so that

$$\int_G f(g) dg = \int_{KAN} f(kan) e^{2\rho(\log a)} dk\, da\, dn, \qquad f \in \mathcal{D}(G).$$

Let $dk_M$ denote the $K$-invariant measure on $K/M$ of total measure 1 which satisfies

$$\int_K f(k) dk = \int_{K/M} \left( \int_M f(km) dm \right) dk_M, \qquad f \in C(K).$$

Let $dg_K = dx$ denote the $G$-invariant measure on $G/K$ given by

$$\int_G f(g) dg = \int_{G/K} \left( \int_K f(gk) dk \right) dg_K, \qquad f \in C_c(G).$$

**3. The Schwartz space and the spherical Fourier transform.** In this section we describe some results of Harish-Chandra [4, 5] and S. Helgason [6, 7] in the form suitable for our purpose.

Let $\varphi_\lambda$ be the spherical function given by

$$\varphi_\lambda(g) = \int_K e^{(i\lambda - \rho)(H(gk))} dk, \qquad (g \in G,\ \lambda \in \mathfrak{a}^*).$$

We put $\Xi(g) = \varphi_0(g)$ $(g \in G)$. Each $g \in G$ can be written uniquely in the form $\exp X.\, k$ $(k \in K,\ X \in \mathfrak{p})$. Then we put $\sigma(g) = \|X\|$.

**Definition 1.** Let $\mathcal{C}(S)$ denote the set of all complex-valued $C^\infty$ functions $f$ on $G$ which satisfy the following two conditions:

(i) $f$ is right-invariant under $K$,

(ii) for each $D \in \boldsymbol{D}(G)$ and each integer $q \geqq 0$

$$\tau_{D,q}(f) = \sup_{g \in G} \Xi(g)^{-1}(1+\sigma(g))^q \, |Df(g)| < +\infty.$$

Let $I(G)$ denote the set of all $f \in \mathcal{C}(S)$ which are left-invariant under $K$. Then $\tau_{D,q}$ is a seminorm on $I(G)$ and $\mathcal{C}(S)$. We topologize $I(G)$ and $\mathcal{C}(S)$ by means of the seminorms $\tau_{D,q}$ ($D \in \boldsymbol{D}(G)$, $q \geqq 0$). In this way $I(G)$ and $\mathcal{C}(S)$ become Fréchet spaces. After Harish-Chandra, we call $\mathcal{C}(S)$ the Schwartz space of $S$. Let $\mathcal{J}(\mathfrak{a}^*)$ denote the set of $W$-invariants in $\mathcal{S}(\mathfrak{a}^*)$.

For $f \in I(G)$, its spherical Fourier transform is defined by

$$\tilde{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g)dg, \qquad (\lambda \in \mathfrak{a}^*). \tag{1}$$

The following theorem is due to Harish-Chandra [4, 5] and Helgason [6].

**Theorem 1.** *Let $f$ be any function in $I(G)$. Then*

(i) $$\int_G |f(x)|^2 \, dx = w^{-1} \int_{\mathfrak{a}^*} |\tilde{f}(\lambda)|^2 \, |c(\lambda)|^{-2} \, d\lambda, \tag{2}$$

(ii) $$f(x) = w^{-1} \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\varphi_\lambda(x) \, |c(\lambda)|^{-2} \, d\lambda. \tag{3}$$

(iii) *The mapping $f \to \tilde{f}$ is a topological isomorphism of $I(G)$ onto $\mathcal{J}(\mathfrak{a}^*)$.*

**4. The Fourier transform of Schwartz space.** For any function $f$ in $\mathcal{C}(S)$, we put

$$\tilde{f}(\lambda, kM) = \tilde{f}_\lambda(kM) = \int_{AN} f(kan)e^{(-i\lambda+\rho)(\log a)} da dn, \tag{4}$$

$\lambda \in \mathfrak{a}^*$ and $kM \in K/M$. If we take into account Theorem B in [1], it can be shown without difficulty that the integral in the right-hand side of (4) converges for each $\lambda$ and $kM$. We call the mapping $f \to \tilde{f}$ the Fourier transform.

**Remark 1.** If we consider only $K$-bi-invariant functions in $\mathcal{C}(S)$ our Fourier transform coincides with the spherical Fourier transform.

**Remark 2.** For any $f \in \mathcal{D}(S)$, our Fourier transform coincides with the one which was defined by S. Helgason in [6].

Using the Iwasawa decomposition, we can extend any function $\phi$ on $\mathfrak{a}^* \times (K/M)$ to a function on $\mathfrak{a}^* \times G$ by

$$\phi(\lambda, x) = e^{(i\lambda-\rho)(H(x))}\phi(\lambda, \kappa(x)M), \qquad x \in G. \tag{5}$$

We also write $\phi(\lambda, x) = \phi_\lambda(x)$. We put

$$\check{\phi}_\lambda(x) = \int_K \phi_\lambda(xk)dk \tag{6}$$

and call the mapping $\phi_\lambda \to \check{\phi}_\lambda$ the dual Radon transform.

Now we define the Schwartz space of $\mathfrak{a}^* \times (K/M)$ as follows.

**Definition 2.** Let $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ denote the set of all complex-valued $C^\infty$ functions $\phi$ on $\mathfrak{a}^*\times(K/M)$ which satisfy the following condition: for each $E\in\boldsymbol{D}(\mathfrak{a}^*)$, $u\in\boldsymbol{D}(K/M)$ and each integer $r\geqq 0$

$$\nu_{E,u,r}(\phi)=\sup_{(\lambda,kM)\in\mathfrak{a}^*\times(K/M)}(1+\|\lambda\|)^r\,|(Eu\phi)(\lambda,kM)|<\infty.$$

Then $\nu_{E,u,r}$ is a seminorm on $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ and the collection of these seminorms, for all $E\in\boldsymbol{D}(\mathfrak{a}^*)$, $u\in\boldsymbol{D}(K/M)$ and integers $r\geqq 0$, defines a topology on $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ in the usual way so that $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ becomes a Fréchet space. And we obtain the following theorems.

**Theorem 2.** *Let $f$ be any function in $\mathcal{C}(S)$. Then $\tilde{f}\in\mathcal{C}(\mathfrak{a}^*\times(K/M))$ and*

$$\check{\tilde{f}}_{s\lambda}=\check{\tilde{f}}_{\lambda} \tag{7}$$

*holds for all $\lambda\in\mathfrak{a}^*$ and $s\in W$.*

**Theorem 3.** *Let $f$ be any function in $\mathcal{C}(S)$. Then*

(i) $\int_G |f(x)|^2\,dx = w^{-1}\int_{\mathfrak{a}^*\times(K/M)} |\tilde{f}(\lambda,kM)|^2\,|c(\lambda)|^{-2}\,dk_M d\lambda,$

(ii) $f(x)=w^{-1}\int_{\mathfrak{a}^*}\check{\tilde{f}}(\lambda,x)\,|c(\lambda)|^{-2}\,d\lambda.$

**Remark 3.** Theorem 3 was proved by S. Helgason [7] for the functions in $\mathcal{D}(S)$.

**Remark 4.** Gel'fand-Graev [3] studied the Paley-Wiener theorem for the Fourier transform on complex semisimple Lie groups $G$. But one should remark that their definition of the space of the infinitely differentiable rapidly decreasing functions is different from that of the Schwartz space $\mathcal{C}(G/K)$.

**5. The image of the Fourier transform.** In order to give a characterization of the image of $\mathcal{C}(S)$ by the Fourier transform we consider the following function space.

**Defintion 3.** Let $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$ denote the set of all functions $\phi$ in $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ which satisfy the following condition: $\check{\phi}_{s\lambda}=\check{\phi}_{\lambda}$ for all $\lambda\in\mathfrak{a}^*$ and $s\in W$.

We consider $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$ with the relative topology induced from $\mathcal{C}(\mathfrak{a}^*\times(K/M))$. Then we can prove the following result.

**Theorem 4.** *For each $\phi\in\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$, put*

$$f_\phi(x)=w^{-1}\int_{\mathfrak{a}^*\times(K/M)}\phi(\lambda,kM)e^{-(i\lambda+\rho)(H(x^{-1}k))}\,|c(\lambda)|^{-2}\,dk_M d\lambda. \tag{8}$$

*Then $f_\phi\in\mathcal{C}(S)$ and the mapping $\phi\to f_\phi$ is a one-to-one continuous linear mapping of $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$ into $\mathcal{C}(S)$.*

Now we can state our main theorem as follows.

**Theorem 5.** *The mapping $f\to\tilde{f}$ defined by (4) is a linear topological isomorphism of $\mathcal{C}(S)$ onto $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$.*

Since $\mathcal{D}(S)\subset\mathcal{C}(S)$, $\mathcal{C}(S)$ is dense in $L^2(S)$ in the $L^2$-topology. Denoting $\mathcal{C}(\mathfrak{a}^*_+\times(K/M))$ the set of all restrictions $\phi|_{\mathfrak{a}^*_+\times(K/M)}$ of $\phi$ in

$\mathcal{C}(\mathfrak{a}^* \times (K/M))$ to $\mathfrak{a}_+^* \times (K/M)$, it is clear that $\mathcal{C}(\mathfrak{a}_+^* \times (K/M))$ is dense in $L^2(\mathfrak{a}_+^* \times (K/M))$ in the $L^2$-topology and that the composition of the Fourier transform and the restriction is an isometry of $\mathcal{C}(S)$ onto $\mathcal{C}(\mathfrak{a}_+^* \times (K/M))$. Hence we obtain the following result.

**Corollary** (S. Helgason). *The Fourier transform $f \to \tilde{f}$ defined by* (4) *extends to an isometry of $L^2(S)$ onto $L^2(\mathfrak{a}_+^* \times (K/M))$ (with the measure $|c(\lambda)|^{-2}\, d\lambda dk_M$ on $\mathfrak{a}_+^* \times (K/M)$).*

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