58. *Studies on Holonomic Quantum Fields.* V

By Mikio SATO, Tetsuji MIWA, and Michio JIMBO
Research Institute for Mathematical Sciences, Kyoto University

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This is a continuation of the series of our notes [1].

Here we shall give a summary of the theory of Clifford group. As for details see [2]. We remark that we have changed the definition of T_g and nr (g) which was given in [1].

1. Norms and rotations. Let W be an N dimensional vector space over $\boldsymbol{C}$. We set $W^*=\mathrm{Hom}_{\boldsymbol{C}}(W,\boldsymbol{C})=\{\eta\,|\,\eta: W\to\boldsymbol{C}, w\mapsto\eta(w)\}$. Let $\Lambda(W)=\bigoplus_{\mu=0}^{N}\Lambda^{\mu}(W)$ denote the exterior algebra over W. We denote by δ the linear homomorphism $\delta: W^*\to\mathrm{End}_{\boldsymbol{C}}(\Lambda(W))$, $\eta\mapsto\delta_\eta$ which satisfies $\delta_\eta(1)=0$ and $\delta_\eta(wa)=\eta(w)a-w\delta_\eta(a)$ for $w\in W$ and $a\in\Lambda(W)$. Let κ be an element of $\mathrm{Hom}_{\boldsymbol{C}}(W,W^*)$ such that $\iota\underset{\mathrm{def}}{=}\kappa+{}^t\kappa$ is invertible. An orthogonal structure is introduced to W by the inner product $\langle w,w'\rangle=\iota(w)(w')=\iota(w')(w)$. We denote by $A(W)$ the Clifford algebra over the orthogonal space W thus obtained.

There exists a unique linear isomorphism

(1.1) $$\mathrm{Nr}_\kappa: A(W)\to\Lambda(W),\qquad a\mapsto\mathrm{Nr}_\kappa(a)$$

which satisfies $\mathrm{Nr}_\kappa(1)=1$ and

(1.2) $$\mathrm{Nr}_\kappa(wa)=w\,\mathrm{Nr}_\kappa(a)+\delta_{\kappa(w)}(\mathrm{Nr}_\kappa(a)).$$

We call $\mathrm{Nr}_\kappa(a)$ the κ-norm of a. The constant term of $\mathrm{Nr}_\kappa(a)$ is called the κ-expectation value and is denoted by $\langle a\rangle_\kappa$.

There exists a unique automorphism $a\mapsto\varepsilon(a)$ (resp. anti-automorphism $a\mapsto a^*$) of $A(W)$ characterized by $\varepsilon(w)=-w$ (resp. $w^*=w$) for $w\in W$. We denote by $G(W)$ the Clifford group $\{g\in A(W)\,|\,{}^{\exists}g^{-1}\in A(W), gW\varepsilon(g)^{-1}=W\}$. We denote by T the group homomorphism $T: G(W)\to O(W)$, $g\mapsto T_g$ defined by $T_g(w)=gw\varepsilon(g)^{-1}$ for $w\in W$. Then we have the following exact sequence.

(1.3) $$1\longrightarrow\mathrm{GL}(1,\boldsymbol{C})\xrightarrow{\mathrm{id.}}G(W)\xrightarrow{T}O(W)\longrightarrow 1.$$

A group homomorphism $\mathrm{nr}: G(W)\to\mathrm{GL}(1,\boldsymbol{C})$, $g\mapsto\mathrm{nr}(g)$ is defined by $\mathrm{nr}(g)=g\varepsilon(g)^*$, which is called the spinorial norm of g.

In what follows we shall adopt the following identifications: $\mathrm{Hom}_{\boldsymbol{C}}(W_1\otimes_{\boldsymbol{C}}W_2,\boldsymbol{C})\cong W_2^*\otimes_{\boldsymbol{C}}W_1^*\cong\mathrm{Hom}_{\boldsymbol{C}}(W_1,W_2^*)$.

If $g\in G(W)$, we have

(1.4) $$\langle g\rangle_\kappa^2=\mathrm{nr}(g)\det((\kappa T_g+{}^t\kappa)\iota^{-1}).$$

If $\langle g\rangle_\kappa\neq 0$, we have

(1.5) $$\mathrm{Nr}_\kappa(g)=\langle g\rangle_\kappa\exp(\rho_g/2)$$

with $\rho_g=(T_g-1)(\kappa T_g+{}^t\kappa)^{-1}\in \Lambda^2(W)\subset W\otimes_C W\cong \mathrm{Hom}_C(W^*, W)$. If $\langle g\rangle_\kappa=0$, then $\mathrm{Ker}\,(\iota^{-1t}\kappa+T_g\iota^{-1}\kappa)\neq 0$. Take a generic element w of W and set $g'=wg$. Then the following conditions i) and ii) for $w_1\in W$ are equivalent;

i) $(\iota^{-1t}\kappa+T_{g'}\iota^{-1}\kappa)(w_1)=0$,

ii) $\begin{cases}(\iota^{-1t}\kappa+T_g\iota^{-1}\kappa)(w_1)=0,\\ \langle w, \iota^{-1t}\kappa(w_1)\rangle=0.\end{cases}$

Moreover we have $\mathrm{Nr}_\kappa(g)=w_1\,\mathrm{Nr}_\kappa(g')$, where w_1 is any element of W satisfying $(\iota^{-1t}\kappa+T_g\iota^{-1}\kappa)(w_1)=0$ and $\langle w, \iota^{-1t}\kappa(w_1)\rangle=1$. Thus the norm of g is of the following form.

(1.6) $$\mathrm{Nr}_\kappa(g)=cw_1\cdots w_k \exp(\rho_g/2)$$

where $c\in \boldsymbol{C}$, $\sum_{j=1}^k \boldsymbol{C}w_j=\mathrm{Ker}\,(\iota^{-1t}\kappa+T_g\iota^{-1}\kappa)$ and $\rho_g\in\Lambda^2(W)$.

Conversely, assume that g is given by (1.6). We set $\mathrm{Nr}_\kappa(g_1)=c\exp(\rho_g/2)$, $W_g=\sum_{j=1}^k \boldsymbol{C}w_j$ and denote by i_g the natural inclusion $i_g: W_g\rightarrow W$. Then we have

(1.7) $$\mathrm{nr}\,(g_1)=\langle g_1\rangle_\kappa^2 \det(1+{}^t\kappa\rho_g).$$

Now assume that $\mathrm{nr}\,(g_1)\neq 0$. Then g_1 belongs to $G(W)$ and we have

(1.8) $$T_{g_1}=(1-\rho_g\kappa)^{-1}(1+\rho_g{}^t\kappa).$$

Moreover we have

(1.9) $$\mathrm{nr}\,(g)=(\det_{(w_1,\dots,w_k)} {}^t i_g(1-\kappa\rho_g)^{-1}\kappa i_g)\,\mathrm{nr}\,(g_1).$$

Here $\det_{(w_1,\dots,w_k)} {}^t i_g(1-\kappa\rho_g)^{-1}\kappa i_g$ means the determinant of the matrix representation of ${}^t i_g(1-\kappa\rho_g)^{-1}\kappa i_g$ with respect to the basis $(w_1,\cdots,w_k)$ and its dual basis. If $\mathrm{nr}\,(g)\neq 0$, g belongs to $G(W)$ and we have

(1.10) $$T_g=T_{g_1}-(1-\rho_g\kappa)^{-1}i_g[{}^t i_g(1-\kappa\rho_g)^{-1}\kappa i_g]^{-1t}i_g(1-\kappa\rho_g)^{-1}\iota.$$

2. The closure of $G(W)$. Let G^k denote the subset $\{cw_1\cdots w_k \exp(\rho/2)\,|\,c\in\boldsymbol{C},\ w_1,\cdots,w_k\in W \text{ and } \rho\in\Lambda^2(W)\}$ of $\Lambda(W)$, and set $G=\bigcup_{k=0}^N G^k$. We also set $\Lambda^+(W)=\bigoplus_{k:\text{ even}}\Lambda^k(W)$, $\Lambda^-(W)=\bigoplus_{k:\text{ odd}}\Lambda^k(W)$ and $G^\pm=G\cap\Lambda^\pm(W)$.

G is closed in $\Lambda(W)$. $P(G^\pm)=(G^\pm-\{0\})/\mathrm{GL}\,(1,\boldsymbol{C})$ is a non-singular projective variety in $P(\Lambda^\pm(W))$ of $(1/2)N(N-1)$ dimensions. $\{P(G^k)\}$ $(k=0,1,\cdots,N)$ gives a stratification of $P(G)$. $P(G^k)$ is a fiber bundle over $M_{N,k}(\boldsymbol{C})$ with the fiber $\Lambda^2(\boldsymbol{C}^{N-k})$. Here we denote by $M_{N,k}(\boldsymbol{C})$ the Grassmann manifold consisting of k dimensional subspaces in $\boldsymbol{C}^N$.

In particular, the closure $\bar{G}(W)$ of $G(W)$ coincides with $\mathrm{Nr}_\kappa^{-1}(\{cw_1\cdots w_k\exp(\rho/2)\,|\,c\in\boldsymbol{C},\ w_1,\cdots,w_k\in W,\ \rho\in\Lambda^2(W) \text{ and } k=0,1,\cdots,N\})$.

Let κ_0 denote the linear homomorphism $\kappa_0: W\rightarrow W^*$ such that $2\kappa_0(w)(w')=\langle w, w'\rangle$. We denote by σ^μ the projection $A(W)\xrightarrow{\mathrm{Nr}_{\kappa_0}}\Lambda(W)=\bigoplus_{\nu=0}^N\Lambda^\nu(W)\xrightarrow{\text{projection}}\Lambda^\mu(W)\xrightarrow{\text{inclusion}}\Lambda(W)\xrightarrow{\mathrm{Nr}_{\kappa_0}^{-1}}A(W)$. For an element $a\in A(W)$ we define

(2.1) $$\sigma_t(a)=\sum_{\mu=0}^N (1+t)^{(N-\mu)/2}(1-t)^{\mu/2}\sigma^\mu(a).$$

If $g \in \overline{G}(W)$, $\sigma_t(g)$ belongs to $\overline{G}(W)$. If $g \in G(W)$, we have

(2.2) $$\mathrm{nr}\,(\sigma_t(g)) \det T_g = \mathrm{nr}\,(g) \det (t + T_g).$$

$\sigma_t(g)$ belongs to $G(W)$ if and only if $\det(1 + T_g t) \neq 0$, in which case we have

(2.3) $$T_{\sigma_t(g)} = (T_g + t)/(1 + T_g t).$$

Note that setting $t = 1$ we have

(2.4) $$(\mathrm{trace}\, g)^2 \det T_g = \mathrm{nr}\,(g) \det (1 + T_g).$$

We adopt the normalization of trace in $A(W)$ so that trace $1 = 2^{N/2}$.

There is a one to one correspondence between κ satisfying $\kappa(w)(w') + \kappa(w')(w) = \langle w, w' \rangle$ and $g \in \overline{G}(W)$ satisfying trace $g = 1$. In fact, the correspondence is given by $\langle a \rangle_\kappa = \mathrm{trace}\, ga$.

3. Transformation law and product. Take a basis $(v_1, \cdots, v_N)$ of W and its dual basis $(v_1^*, \cdots, v_N^*)$ of W^*. We denote by K and J the matrix $(\langle v_\mu v_\nu \rangle)_{\mu,\nu=1,\dots,N}$ and $(\langle v_\mu, v_\nu \rangle)_{\mu,\nu=1,\dots,N}$, respectively. The matrix representations of κ and ι with respect to the above basis read ${}^tK$ and J, respectively.

Let $g \in \overline{G}(W)$ be given by $\mathrm{Nr}_\kappa(g) = c w_1 \cdots w_k \exp(\rho/2)$. Set $\boldsymbol{r} = \begin{pmatrix} c_{1,1} & \cdots & c_{k,1} \\ \vdots & & \vdots \\ c_{1,N} & \cdots & c_{k,N} \end{pmatrix}$ where $w_j = \sum_{\mu=1}^N v_\mu c_{j,\mu}$, and set $R = (R_{\mu\nu})_{\mu,\nu=1,\dots,N}$ where $\rho = \sum_{\mu,\nu=1}^N R_{\mu\nu} v_\mu v_\nu$. Let e_μ denote the N component column vector $(\delta_{\mu\nu})_{\nu=1,\dots,N}$.

If we write $\mathrm{Nr}_\kappa(g) = \sum_{m=0}^N 1/m! \sum_{\mu_1,\dots,\mu_m=1}^N \rho_m(\mu_1, \cdots, \mu_m) v_{\mu_m} \cdots v_{\mu_1}$, the coefficient $\rho_m(\mu_1, \cdots, \mu_m)$ is given by

$$\rho_m(\mu_1, \cdots, \mu_m) = \mathrm{Pfaffian}\begin{pmatrix} {}^t\boldsymbol{e} & \\ & {}^t\boldsymbol{r} \end{pmatrix}\begin{pmatrix} -R & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \boldsymbol{e} & \\ & \boldsymbol{r} \end{pmatrix}$$

(3.1) $$= (-1)^{(m+k)/2}\, \mathrm{Pfaffian}\left[\begin{array}{cc|cc} & & {}^t\boldsymbol{e} & \\ & & & {}^t\boldsymbol{r} \\ \hline -\boldsymbol{e} & & & 1 \\ & -\boldsymbol{r} & -1 & R \end{array}\right] / \mathrm{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

where $\boldsymbol{e} = (e_{\mu_1}, \cdots, e_{\mu_m})$.

Now let κ and κ' be such that $\kappa + {}^t\kappa = \kappa' + {}^t\kappa' = \iota$ and let K and K' be the corresponding matrices, respectively. We set $\mathrm{Nr}_\kappa(g_1) = c \exp(\rho/2)$. Then we have

$$\langle g_1 \rangle_{\kappa'} = \langle g_1 \rangle_\kappa (\det(1 - (K' - K)R))^{1/2}$$

(3.2) $$= \langle g_1 \rangle_\kappa\, \mathrm{Pfaffian}\begin{pmatrix} -(K' - K) & 1 \\ -1 & R \end{pmatrix} / \mathrm{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

If $\langle g_1 \rangle_{\kappa'} \neq 0$, we have

(3.3) $$\mathrm{Nr}_{\kappa'}(g_1) = \langle g_1 \rangle_{\kappa'} \exp(\rho'/2)$$

where $\rho' = \sum_{\mu,\nu=1}^N R'_{\mu\nu} v_\mu v_\nu$ with $R' = R(1 - (K' - K)R)^{-1}$. Moreover if we write $\mathrm{Nr}_{\kappa'}(g) = \sum_{m=0}^N 1/m! \sum_{\mu_1,\dots,\mu_m=1}^N \rho'_m(\mu_1, \cdots, \mu_m) v_{\mu_m} \cdots v_{\mu_1}$, the coefficient $\rho'_m(\mu_1, \cdots, \mu_m)$ is given by

$$\rho'_m(\mu_1,\cdots,\mu_m)=(-)^{(m+k)/2}\langle g\rangle_\kappa$$

(3.4)
$$\times\,\mathrm{Pfaffian}\left[\begin{array}{cc|cc} & & {}^t e & \\ & & & {}^t r \\ \hline -e & & -(K'-K) & 1 \\ & -r & -1 & R \end{array}\right]\Big/\mathrm{Pfaffian}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}.$$

Next we shall give formulas for products of elements in $\overline{G}(W)$. If $w\in W$ and $\mathrm{Nr}_\kappa(g)=cw_1\cdots w_k\exp(\rho/2)$, we have

(3.5)
$$\mathrm{Nr}_\kappa(wg)=c\Big(\sum_{j=1}^{k}(-)^{j-1}w_1\cdots w_{j-1}\langle ww_j\rangle_\kappa w_{j+1}\cdots w_k + \tilde{w}w_1\cdots w_k\Big)\exp(\rho/2)$$

where $\tilde{w}=(1-\rho\kappa)(w)$.

Let $W^{(\nu)}$ $(\nu=1,\cdots,n)$ be copies of W. Let Λ denote an $n\times n$ symmetric matrix $(\lambda_{\mu\nu})_{\mu,\nu=1,\cdots,n}$ with $\lambda_{\nu\nu}=1$ $(\nu=1,\cdots,n)$. Let $W(\Lambda)$ denote the vector space $\bigoplus_{\nu=1}^n W^{(\nu)}$ equipped with the inner product $\langle(w^{(1)},\cdots,w^{(n)}),(w'^{(1)},\cdots,w'^{(n)})\rangle_\Lambda=\sum_{\mu,\nu=1}^n\lambda_{\mu\nu}\langle w^{(\mu)},w'^{(\nu)}\rangle$. If $\det\Lambda\neq0$, $W(\Lambda)$ is an orthogonal space. Let κ_Λ denote an element of $\mathrm{Hom}_{\boldsymbol{C}}(W(\Lambda),W(\Lambda)^*)$ given by

$$\kappa_\Lambda((w^{(1)},\cdots,w^{(n)}))((w'^{(1)},\cdots,w'^{(n)}))=\sum_{\mu,\nu=1}^n\lambda_{\mu\nu}\kappa(w^{(\mu)})(w'^{(\nu)}).$$

Let $g^{(\nu)}$ be an element of $\overline{G}(W^{(\nu)})\subset\overline{G}(W(\Lambda))$ given by $\mathrm{Nr}_\kappa(g^{(\nu)})=c^{(\nu)}w_1^{(\nu)}\cdots w_{k^{(\nu)}}^{(\nu)}\exp(\rho^{(\nu)}/2)$, with $\rho^{(\nu)}=\sum_{j,l=1}^n R_{jl}^{(\nu)}v_j^{(\nu)}v_l^{(\nu)}$. We set $\mathrm{Nr}_\kappa(g_1^{(\nu)})=c^{(\nu)}\exp(\rho^{(\nu)}/2)$. Let $c_j^{(\nu)}$ denote the column vector ${}^t(c_{j,1}^{(\nu)},\cdots,c_{j,N}^{(\nu)})$ where $w_j^{(\nu)}=\sum_{m=1}^N v_m^{(\nu)}c_{j,m}^{(\nu)}$, and let r be an $Nn\times k$ matrix $\begin{bmatrix} c_1^{(1)}\cdots c_{k^{(1)}}^{(1)} & & \\ & \ddots & \\ & & c_1^{(n)}\cdots c_{k^{(n)}}^{(n)}\end{bmatrix}$, where $k=\sum_{\mu=1}^n k^{(\mu)}$. Let $(\hat{v}_1,\cdots,\hat{v}_{Nn})$ denote the basis $(v_1^{(1)},\cdots,v_N^{(1)},\cdots,v_1^{(n)},\cdots,v_N^{(n)})$ and let $\hat{e}_1,\cdots,\hat{e}_{Nn}$ denote the Nn component column vectors, $\begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\ \vdots\\0\end{bmatrix}$, etc., respectively. Let R and $A(\Lambda)$ denote an $Nn\times Nn$ skew-symmetric matrix

$$\begin{bmatrix}R^{(1)} & & \\ & \ddots & \\ & & R^{(n)}\end{bmatrix}\ \text{and}\ \begin{bmatrix}0 & \lambda_{12}K & \cdots & \lambda_{1n}K\\ -\lambda_{21}{}^tK & 0 & & \vdots\\ \vdots & & \ddots & \lambda_{n-1n}K\\ -\lambda_{n1}{}^tK & \cdots & -\lambda_{nn-1}{}^tK & 0\end{bmatrix},$$ respectively. Then we have

(3.6)
$$\langle g_1^{(1)}\cdots g_1^{(n)}\rangle_{\kappa_\Lambda}=\langle g_1^{(1)}\rangle_\kappa\cdots\langle g_1^{(n)}\rangle_\kappa(\det(1-A(\Lambda)R))^{1/2}$$
$$=\langle g_1^{(1)}\rangle_\kappa\cdots\langle g_1^{(n)}\rangle_\kappa\,\mathrm{Pfaffian}\begin{pmatrix}-A(\Lambda) & 1\\ -1 & R\end{pmatrix}\Big/\mathrm{Pfaffian}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}.$$

If $\langle g_1^{(1)}\cdots g_1^{(n)}\rangle_{\kappa_\Lambda}\neq0$, we have

(3.7) $$\mathrm{Nr}_{\kappa_\Lambda}(g_1^{(1)}\cdots g_1^{(n)})=\langle g_1^{(1)}\cdots g_1^{(n)}\rangle_{\kappa_\Lambda}\exp(\rho(\Lambda)/2)$$

where $\rho(\Lambda)=\sum_{\mu,\nu=1}^{Nn} R(\Lambda)_{\mu\nu}\hat{v}_\mu\hat{v}_\nu$ with $R(\Lambda)=R(1-A(\Lambda)R)^{-1}$. If we write $\mathrm{Nr}_{\kappa_\Lambda}(g^{(1)}\cdots g^{(n)})=\sum_{m=0}^{Nn}1/m!\sum_{\mu_1,\dots,\mu_m=1}^{Nn}\rho_m(\mu_1,\cdots,\mu_m)\hat{v}_{\mu_m}\cdots\hat{v}_{\mu_1}$, the coefficient $\rho_m(\mu_1,\cdots,\mu_m)$ is given by

(3.8) $$\rho_m(\mu_1,\cdots,\mu_m)=(-)^{(m+k)/2}\langle g_1^{(1)}\rangle_\kappa\cdots\langle g_1^{(n)}\rangle_\kappa \left[\begin{array}{cc|cc} & & {}^te & \\ & & & {}^tr \\ \hline -e & & -A(\Lambda) & 1 \\ & -r & -1 & R \end{array}\right]$$

where $e=(e_{\mu_1},\cdots,e_{\mu_m})$. If $\langle g^{(1)}\cdots g^{(n)}\rangle_{\kappa_\Lambda}\neq 0$, we have

(3.9) $$\rho_m(\mu_1,\cdots,\mu_m)=\langle g^{(1)}\cdots g^{(n)}\rangle_{\kappa_\Lambda} \times\mathrm{Pfaffian}\begin{pmatrix}{}^te & \\ & {}^tr\end{pmatrix}\begin{pmatrix}-R(1-A(\Lambda)R)^{-1} & (1-RA(\Lambda))^{-1}\\ -(1-A(\Lambda)R)^{-1} & (1-A(\Lambda)R)^{-1}A(\Lambda)\end{pmatrix}\begin{pmatrix}e & \\ & r\end{pmatrix}.$$

4. The extended Clifford group. Since we have not expounded this subject in [2], here we shall explain it in detail.

Let us consider the orthogonal space $C\oplus W$ equipped with the inner product $\langle c+w, c'+w'\rangle=-\{(c+w)\varepsilon(c'+w')+(c'+w')\varepsilon(c+w)\}=-2cc'+\langle w,w'\rangle$. Let $G_{\mathrm{ext}}(W)$ denote the extended Clifford group $\{g\in A(W)\,|\,{}^\exists\varepsilon(g)^{-1}, g(C\oplus W)\varepsilon(g)^{-1}=C\oplus W\}$. We denote by T_g the linear transformation of $C\oplus W$ induced by g, $T_g: c+w\mapsto g(c+w)\varepsilon(g)^{-1}$. $c+w\in C\oplus W$ belongs to $G_{\mathrm{ext}}(W)$ if and only if $-c^2+w^2\neq 0$, and we have

(4.1) $$T_{c+w}(c'+w')=-(c'-w')+\{(-2cc'-\langle w,w'\rangle)/(-c^2+w^2)\}(c+w).$$

If we denote by $\hat{T}_{c+w}$ the reflection in $C\oplus W$ with respect to the hyperplane $\{c'+w'\in C\oplus W\,|\,\langle c+w, c'+w'\rangle=0\}$, then (4.1) reads

(4.2) $$T_{c+w}=-\hat{T}_{c+w}\circ\varepsilon=-\varepsilon\circ\hat{T}_{c-w}.$$

This implies that any element of $G_{\mathrm{ext}}(W)$ is of the form $(c_1+w_1)\cdots(c_k+w_k)$ with $c_j+w_j\in(C+W)\cap G_{\mathrm{ext}}(W)$. The following exact sequence is valid

(4.3) $$1\to GL(1,C)\to G_{\mathrm{ext}}(W)\to SO(C\oplus W)\to 1.$$

Let $W_{\mathrm{ext}}\underset{\mathrm{def}}{=}Cw_0\oplus W$ be an orthogonal space, where w_0 satisfies the following: $w_0^2=-1$, $\langle w, w_0\rangle=0$ for any $w\in W$. The theory of the extended Clifford group is reduced to that of $G^+(W_{\mathrm{ext}})=\{g\in G(W_{\mathrm{ext}})\,|\,\varepsilon(g)=g\}$. Firstly $F_{C\oplus W}: C\oplus W\to W_{\mathrm{ext}}$, $c+w\mapsto cw_0+w$ is an isomorphism. We also denote by $F_{A(W)}$ the isomorphism $A(W)\to A^+(W_{\mathrm{ext}})$, $a^++a^-\mapsto a^++w_0a^-$. Note that $F_{C\oplus W}(c+w)=F_{A(W)}(c+w)w_0$. We have $F_{A(W)}(\varepsilon(a)^*)=\varepsilon(F_{A(W)}(a))^*$ and $\mathrm{nr}(g)\underset{\mathrm{def}}{=}g\varepsilon(g)^*=\mathrm{nr}(F_{A(W)}(g))$ for $g\in G_{\mathrm{ext}}(W)$. Moreover we have for $g\in G_{\mathrm{ext}}(W)$

(4.4) $$F_{C\oplus W}\circ T_g=T_{F_{A(W)}(g)}\circ F_{C\oplus W},$$

and the exact sequence (4.3) isomorphically is transformed into

(4.5) $$1\to GL(1,C)\to G^+(W_{\mathrm{ext}})\to SO(W_{\mathrm{ext}})\to 1.$$

Let κ be an element of $\mathrm{Hom}_C(W, W^*)$, and define $\kappa_{\mathrm{ext}}\in\mathrm{Hom}_C(W_{\mathrm{ext}}, W_{\mathrm{ext}}^*)$ by $\langle ww'\rangle_{\kappa_{\mathrm{ext}}}=\langle ww'\rangle_\kappa$, $\langle ww_0\rangle_{\kappa_{\mathrm{ext}}}=0$ for $w, w'\in W$. If we denote by

$F_{\Lambda(W)}$ the isomorphism $\Lambda(W) \rightarrow \Lambda^+(W_{\mathrm{ext}})$, $a^+ + a^- \mapsto a^+ + w_0 a^-$, then we have

(4.6) $$F_{\Lambda(W)} \circ \mathrm{Nr}_\kappa = \mathrm{Nr}_{\kappa\,\mathrm{ext}} \circ F_{\Lambda(W)}.$$

(4.4) and (4.6) provide us with a means to compute the norm of an element of $G_{\mathrm{ext}}(W)$ and the rotation it induces in $\boldsymbol{C} \oplus W$ from each other. In particular, the closure $\overline{G}_{\mathrm{ext}}(W)$ coincides with $\mathrm{Nr}_\kappa^{-1}(\{c w_1 \cdots w_k \cdot \exp(\rho/2 + w) \mid c \in \boldsymbol{C}, w_1, \cdots, w_k, w \in W, \rho \in \Lambda^2(W)\})$.

References

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[2] ——: RIMS preprint **234** (1977).