56. *Cartan Connection and Characteristic Classes of Foliations*

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1. Introduction. In this note we define characteristic classes of conformal and projective foliations, keeping in mind the strong vanishing theorem of Nishikawa and Sato concerning the Pontrjagin classes of the normal bundles [5], and describe the relationship of them with those of smooth foliations defined by Bott and Haefliger [1] and those of Riemannian foliations due to Lazarov and Pasternack [4] (see also Kamber and Tondeur [2] for a general theory). The main point of the construction is the use of Cartan connection, which also yields new characteristic classes for Riemannian foliations.

2. Riemannian case. Let F be a codimension n Riemannian foliation on a smooth manifold M and let $O(F)$ be the orthonormal frame bundle of F. If we denote $\mathfrak{e}(n)$ for the Lie algebra of the group of Euclidean motions on $\boldsymbol{R}^n$, then the canonical form and the unique Riemannian connection form on $O(F)$ define a d.g.a. map

$$\phi\colon W(\mathfrak{e}(n))\rightarrow\Omega^*(O(F))$$

where $W(\mathfrak{e}(n))$ is the Weil algebra of $\mathfrak{e}(n)$ and $\Omega^*(O(F))$ is the de Rham complex of $O(F)$. Now ϕ has a kernel I: the ideal generated by the facts (i) torsionfree-ness of the Riemannian connection (ii) the first Bianchi's identity and (iii) the curvature form is horizontal. Let $\tilde{W}(\mathfrak{e}(n))=W(\mathfrak{e}(n))/I$ and assume that the normal bundle of F is trivialized by a cross section $s\colon M\rightarrow O(F)$, then we obtain

$$\phi^*\colon H^*(\tilde{W}(\mathfrak{e}(n)))\longrightarrow H^*_{DR}(O(F))\xrightarrow{s^*}H^*_{DR}(M).$$

Since this construction is functorial, we have

$$\phi^*\colon H^*(\tilde{W}(\mathfrak{e}(n)))\rightarrow H^*(BR\bar{\Gamma}_n;\boldsymbol{R}),$$

where $BR\bar{\Gamma}_n$ is the classifying space for codimension n Riemannian Haefliger structures with trivial normal bundles. As for the cohomology of $\tilde{W}(\mathfrak{e}(n))$, we have

Theorem 1. $H^*(\tilde{W}(\mathfrak{e}(n)))=H^*(\tilde{W}(\mathfrak{so}(n)))+\sum_{\substack{0\leqq p<n\\ p:\text{even}}} r_pH^*(\mathfrak{so}(n))$.

Here $H^*(\tilde{W}(\mathfrak{so}(n)))$ is the cohomology of the truncated Weil algebra of $\mathfrak{so}(n)$ studied by Kamber and Tondeur [2] and is the same as the characteristic classes defined by Lazarov and Pasternack. $\phi^*(r_0)$ is the

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volume form and $\phi^*(r_p)=R_p$ volume form $(0<p<n)$, where R_p is the "p-th scalar curvature" which is defined by averaging the p-th sectional curvature γ_p of Allendoerfer (cf. Thorpe [6]). The classes of the second term of Theorem 1 can be varied continuously and independently. Thus we obtain

Theorem 2. *Let* $\dim H^k(\mathfrak{so}(n))=d$. *Then there is a surjective homomorphism*

$$H_{n+k}(BR\Gamma_n\,;\boldsymbol{Z})\to \boldsymbol{R}^{d[(n+1)/2]}\to 0.$$

3. Conformal and projective cases. Let F be a conformal (resp. projective) foliation on a smooth manifold M (cf. [5]) and let $J_2(F)$ be the 2-jet bundle of F. If $L=O(n+1,1)$ (resp. $PGL(n\,;\boldsymbol{R})$) acting on the sphere S^n (resp. the real projective space P^n) and if L_0 is the isotropy subgroup at the base point, then $J_2(F)$ has a principal subbundle $P(F)$ with structure group L_0. Now by virtue of a theorem of E. Cartan on the existence and uniqueness of the normal Cartan connection (cf. [3]), we have a d.g.a. map

$$\phi\colon W(\mathfrak{l})\to \Omega^*(P(F))$$

where $\mathfrak{l}$ is the Lie algebra of L. As in the Riemannian case, ϕ has a kernel I: the ideal constructed from the facts (i) torsionfree-ness and normality of Cartan connection (ii) the first Bianchi's identity and (iii) the curvature form is horizontal. If we put $\tilde{W}(\mathfrak{l})=W(\mathfrak{l})/I$, then we have

$$H^*(\tilde{W}(\mathfrak{l}))\to H^*(BS\Gamma_n\,;\boldsymbol{R})$$

where $S=C$ (resp. P) and $BS\Gamma_n$ is the classifying space for conformal (resp. projective) Haefliger structures with trivial normal bundles. Now define two d.g.a.'s CW_n and PW_n by

$$\begin{aligned} CW_n&=\hat{\boldsymbol{R}}[c_1, c_2, c_4, \cdots, c_{n-1}]\otimes E(h_1, h_2, h_4, \cdots, h_{n-1}), && n \text{ odd},\\ &=\hat{\boldsymbol{R}}[c_1, c_2, c_4, \cdots, c_{n-2}, \chi, \bar{\chi}]\otimes E(h_1, h_2, h_4, \cdots, h_{n-2}, h_\chi), && n \text{ even},\\ PW_n&=\hat{\boldsymbol{R}}[c_1, c_2, \cdots, c_n]\otimes E(h_1, h_2, \cdots, h_n), \end{aligned}$$

where $\boldsymbol{R}[c_1, \cdots]$ denotes the polynomial algebra over $\boldsymbol{R}$ in the variables $c_1, \cdots$ with $\deg c_i=2i$, $\deg \chi=\deg \bar{\chi}=n$ and E is the exterior algebra. The differential is given by $dc_i=d\chi=d\bar{\chi}=0$, $dh_i=c_i$, $dh_\chi=\chi$ and $\hat{\boldsymbol{R}}[c_1, \cdots]$ denotes the algebra $\boldsymbol{R}[c_1, \cdots]$ devided by an ideal J defined as follows: We define the "length" l by $l(c_1)=1$, $l(c_{2i})=4i$, $l(\chi)=n/2$, $l(\bar{\chi})=n$. Then J (of the conformal case with odd n and projective case) is the ideal generated by

(i) elements whose length $>n$.

J of the conformal case with even n is the one generated by (i) and

(ii) $c_1\chi$

(iii) $\chi^2-c_1^n$.

with these understood we have

Theorem 3. $H^*(\tilde{W}(\mathfrak{so}(n+1,1)))=H^*(CW_n)$ *and*

$$H^*(\tilde{W}(\mathfrak{sl}(n+1;\boldsymbol{R})))=H^*(PW_n).$$

4. Relations among various characteristic classes. Recall that $H^*(\tilde{W}(\mathfrak{gl}(n;\boldsymbol{R})))$: the cohomology of a certain truncated Weil algebra of $\mathfrak{gl}(n;\boldsymbol{R})$ defines characteristic classes of smooth foliations (cf. [2]). Now we can define various d.g.a. maps among $\tilde{W}(\mathfrak{gl}(n;\boldsymbol{R}))$, $\tilde{W}(\mathfrak{so}(n+1, 1))$, $\tilde{W}(\mathfrak{sl}(n+1;\boldsymbol{R}))$ and $\tilde{W}(\mathfrak{e}(n))$ so that the following diagram is commutative.

$$\begin{tikzcd}
H^*(\tilde{W}(\mathfrak{gl}(n;\boldsymbol{R}))) \arrow[rr] \arrow[dr] \arrow[dd] & & H^*(B\Gamma_n;\boldsymbol{R}) \arrow[dr] \arrow[dd] & \\
 & H^*(\tilde{W}(\mathfrak{so}(n+1,1))) \arrow[rr] \arrow[dd] & & H^*(BC\Gamma_n;\boldsymbol{R}) \arrow[dd] \\
H^*(\tilde{W}(\mathfrak{sl}(n+1;\boldsymbol{R}))) \arrow[rr] \arrow[dr] & & H^*(BP\Gamma_n;\boldsymbol{R}) \arrow[dr] & \\
 & H^*(\tilde{W}(\mathfrak{e}(n))) \arrow[rr] & & H^*(BR\Gamma_n;\boldsymbol{R}).
\end{tikzcd}$$

We can also completely determine the homomorphisms among $H^*(\tilde{W}(\ \ ))$'s. For example, we have

Theorem 4. *The characteristic classes of Riemannian foliations due to Lazarov and Pasternack* [4] *can already be defined in the conformal context and the same is true for the projective case if we exclude the classes containing the Euler form.*

Theorem 5. *As far as the characteristic classes of smooth foliations are concerned, the curvature of the Cartan connection does not play any role on conformal nor projective foliations. In particular, the rigid classes of smooth foliations are all zero on such foliations.*

5. Remarks. (1) Yamato [7] has proved the latter part of Theorem 5 under a condition "local homogenuity".

(2) It might be interesting if one could extend the above results to foliations associated with more general second order G-structures.

(3) In this note, we considered only foliations with trivial normal bundles. However the general case can also be treated.

The detailed description of the materials in this note will appear elsewhere.

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