43. Three-Dimensional Dirichlet Problem for the Helmholtz Equation for an Open Boundary

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1. Let us mean by an open boundary a union of a finite number of simple, smooth, bounded, simply or multiply connected, and two-sided open surfaces in R^3, each of which being bounded by a union of piecewise smooth, simple and closed contours of finite length. It is the purpose of this paper to publish the résumé of the theory of the Dirichlet problem for the Helmholtz equation and for an open boundary which the author has established recently.

In the previous papers [1, 2], the author completed the theory of the two-dimensional Dirichlet problem for an open boundary composed of a union of simple and smooth arcs in a plane. In [1], he depended on a theory of a singular integral equation, whose approach was difficult to extend to apply to the three-dimensional case. However, the series expansion approach taken in [2] was studied carefully so that it can be extended to apply to the three-dimensional problems. The extension will be described in the present paper.

Since the space does not allow a detailed description, only the main results will be given, and the full length paper is expected to appear in some periodical soon.

2. Let S be the above mentioned union of open surfaces, ∂S be it's periphery, and set $S^0 = S - \partial S$. Let us denote points in R^3 by x, y, etc., the distance between x and y by $d(x, y)$, and the distance between x and ∂S by $d(x, \partial S)$. Suppose that $S(\rho)$ and $S^*(\rho)$, where ρ is a positive constant, are defined by $S^*(\rho) = \{x ; x \in S, d(x, \partial S) < \rho\}$ and $S(\rho) = S - S^*(\rho)$, respectively. Assume that the unit vector normal to S is given on S^0. With regards to the directions of the normal, each side of S is called the positive or negative side of it, respectively. If a function $f(x)$ assumes a definite limit as x ($\in R^3 - S$) tends to a point x_0 on S_0 from the positive (negative) side of S, it is said to be continuous on S from the positive (negative) side of it, and the limiting value is denoted by $f^+(x)(f^-(x))$. Let $C(S), L_1(S)$ and $L_2(S)$ be, as usual, the spaces of functions which are continuous on S, integrable on S, and square integrable on S, respectively. Suppose that $T(S)$ is the set of all functions belonging to $L_1(S) \cap C(S(\rho))$ for any $\rho > 0$. Our problem, which will be called Problem D for the sake of brevity, is to find a function $u(x)$

such that $u(x)$ is twice continuously differentiable in R^3-S, that $u(x)$ and it's first order derivatives are continuous on S from the positive as well as negative sides of it, and that it satisfies the following conditions;

(1) $$\Delta u(x) + k^2 u(x) = 0, \qquad x \in R^3 - S,$$

where Δ is the three-dimensional Laplace's operator, and k is a complex-valued constant such that Im $k \leqq 0$;

(2) $$u^{\pm}(x) = \gamma_{\pm}(x), \qquad x \in S,$$

where $\gamma_{\pm}$ are continuous functions given on S;

(3) $$\lim_{R \to \infty} \int_{S(R)} \left| \frac{\partial u(x)}{\partial R} + iku(x) \right|^2 dS_x = 0,$$

where $S(R)$ is a sphere of radius R and an arbitrarily fixed center; and finally

(4) $$\lim_{\rho \to 0} \int_{S^{**}(\rho) - S} \left\{ \left| \frac{\partial u(x)}{\partial n} \right| + |u(x)| \right\} dS_x = 0,$$

where $S^{**}(\rho)$ is defined as follows. In a plane perpendicular to ∂S at a regular point x on ∂S, we draw a circle K of radius ρ and center x. $S^{**}(\rho)$ is the union of surfaces which K generates when x moves along ∂S. $\partial/\partial n$ stands for the differentiation in the direction of the normal of a surface.

(4) is the simplest edge condition (condition of finite energy) which makes the solution unique. However, it is noted [1], that there exist infinitely many edge conditions each of which makes the corresponding solution unique. Moreover, a problem with another edge condition is reduced easily to the present one with condition (4).

To begin with, with help of Green's second identity, we can prove

Theorem 1. *A solution of Problem* D, *if exists, is necessarily represented as*

(5) $$u(x) = \int_S \psi(x, y) \tau(y) dS_y - u_0(x), \qquad x \in R^3 - S,$$

where we have set $\psi(x, y) = e^{-ikd(x,y)}/4\pi d(x, y)$, $\gamma_0(x) = \gamma_-(x) - \gamma_+(x)$, *and*

$$u_0(x) = \int_S \{ \partial \psi(x, y) / \partial n(y) \} \cdot \gamma_0(y) dS_y.$$

In (5), $\tau(y) \in T(S)$, *and it should satisfy the following integral equation of Fredholm of the first kind,*

(6) $$\Psi \tau \equiv \int_S \psi(x, y) \tau(y) dS_y = g(x), \qquad x \in S,$$

where

$$g(x) = \frac{1}{2} \{ \gamma_-(x) + \gamma_+(x) \} + u_0(x).$$

Theorem 2. *Conversely, if functions* $\gamma_0(x)$ *and* $g(x)$ *are given arbitrarily, and if* $\tau(x) \in T(S)$ *is found so as to satisfy* (6), *then,* $u(x)$ *defined by* (5) *in terms of* τ *is the solution of Problem* D *satisfying all*

of the conditions (1)–(4).

These theorems prove Problem D to be equivalent to that of solving for the integral equation (6).

On applying Green's first identity in the domain $D(R)$ bounded by $S(R)$ and S, a solution u of Problem D is proved to satisfy

Lemma 1.

$$(7) \quad \lim_{R \to \infty} \left[\iint_{S(R)} \left\{ \left| \frac{\partial u}{\partial n} \right|^2 + |ku|^2 \right\} dS - 2(\operatorname{Im} k) \int_{D(R)} \{|\nabla u|^2 + |ku|^2\} dV \right]$$
$$= -2 \operatorname{Im} \left[k \int_S \left\{ \gamma_- \overline{\left(\frac{\partial u}{\partial n} \right)^-} + \gamma_+ \overline{\left(\frac{\partial u}{\partial n} \right)^+} \right\} dS \right]$$

where — *denotes the complex conjugate.* As a consequence of Theorem 2 and Lemma 1, we have

Theorem 3. $\Psi\tau=0$ *is equivalent to* $\tau=0$.

This proves the uniqueness of the solution of (6), and hence, of Problem D as well. Ψ is an additive operator mapping $T(S)$ into $C(S)$. (Actually, the range $R(\Psi)$ is proved to be $C(S)$). The inverse operator Ψ^{-1} is certified to exist by Theorem 3, and is proved to be "continuous" in the sense of the following theorem.

Theorem 4. *For arbitrary* $\varepsilon > 0$ *and* $\rho > 0$, *there exists a* $\delta > 0$ *such that* $\|g\|_{C(S)} = \sup_S |g(x)| < \delta$ *implies* $\|\tau\|_{C(S(\rho))} = \sup_{S(\rho)} |\tau(x)| < \varepsilon$.

This theorem is proved by Theorem 2 and Lemma 1.

Let $\{\varphi_n\}$ be a complete system of functions in $L_2(S)$, $u_n(x)$ be defined by $\Psi\varphi_n$, and $U(S)$ be the linear space generated by $\{u_n\}$. Then, with helps of Hahn-Banach's extension theorem and Riesz's theorem, we can prove

Theorem 5. $U(S)$ *is dense in* $C(S)$.

By virtue of this theorem, there exist a sequence $\{g_N\}$, where $g_N(x) = \sum_{n=1}^N c_n u_n(x)$ and c's are constants, which converges to $g(x)$ uniformly on S. This implies that, by Theorem 4, $\tau_N(x) = \sum_{n=1}^N c_n \varphi_n(x)$ converges to a function $\tau(x)$ uniformly on $S(\rho)$ for any $\rho > 0$. As a consequence, we have

Theorem 6. *For a given continuous function* $g(x)$, *there exists a solution* τ *of eq.* (6).

Finally, it is shown that

$$u_N(x) = \sum_{n=1}^N c_n \int_S \psi(x, y)\varphi_n(y) dS_y - u_0(x)$$

is an approximation of the solution of Problem D that converges to it uniformly in wider sense in $R^3 - S$. It is noted that Theorems 4 and 5 hold even if the norm $\|g\|_{C(S)}$ and the space $C(S)$ are replaced by

$$\|g\|_{L^2(S)} = \left\{ \iint_S |g(x)|^2 \, dS_x \right\}^{1/2}$$

and $L_2(S)$, respectively. This result is more useful in the numerical analysis of Problem D.

From the point of view of application, the result obtained here will solve various diffraction problems of acoustic waves. In the following paper, the same problem for the Maxwell's equations will be studied, with the intention to solve diffraction problems of electromagnetic waves. Finally, it is noted that the present theory is easily modified to hold for the theory of the Dirichlet problem for the Laplace (potential) equation.

References

[1] Hayashi, Y.: The Dirichlet problem for the two-dimensional Helmholtz equation for an open boundary. J. Math. Anal. Appl., **44**, 489–530 (1973).
[2] ———: The expansion theory of the Dirichlet problem for the Helmholtz equation for an open boundary (received by J. Math. Anal. Appl., May, 1976).