42. *Studies on Holonomic Quantum Fields. III*

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In this note we report along with [1] the work presented in [2]. Further results along the present line will be given in subsequent papers.

We follow the same notations as in [1] and [3] unless otherwise stated. In this article, along with the 2-dimensional space-time (=Minkowski 2-space) and its complexification, to be denoted by X^{Min} and X^{C} respectively, we also deal with the Euclidean 2-space X^{Euc} consisting of complex Minkowski 2-vectors $x\in X^{C}$ such that $x^0\,(=-ix^2)\in i\boldsymbol{R}$ and $x^1\in\boldsymbol{R}$, i.e. such that $\mp x^{\mp}\;(=(\mp x^0+x^1)/2)$ are complex conjugate to each other; we have $z=-x^-$, $\bar{z}=x^+$, $\partial_z=\partial/\partial z$ and $\partial_{\bar z}=\partial/\partial\bar{z}$.

1. Let W be an orthogonal vector space, and $W=V^{\dagger}\oplus V$ be its decomposition into two holonomic subspaces with basis $(\psi^{\dagger}_{\mu})$ and (ψ_{μ}) as in §2 [3]. V (resp. $V^{\dagger}$) generates maximal left (resp. right) ideal $A(W)V$ (resp. $V^{\dagger}A(W)$) of the Clifford algebra $A(W)$. The quotient modules $A(W)/A(W)V$ and $A(W)/V^{\dagger}A(W)$ are generated by the residue class of 1 modulo $A(W)V$ resp. $V^{\dagger}A(W)$ (which we shall denote by $|\mathrm{vac}\rangle$ and $\langle\mathrm{vac}|$ respectively after physicists' notation) and coincide with $A(V^{\dagger})|\mathrm{vac}\rangle$ and $\langle\mathrm{vac}|A(V)$ since we have $V|\mathrm{vac}\rangle=0$ and $\langle\mathrm{vac}|V^{\dagger}=0$. Otherwise stated, they are respectively spanned by elements of the form $|\nu_n,\cdots,\nu_1\rangle\underset{\mathrm{def}}{=}\psi^{\dagger}_{\nu_n}\cdots\psi^{\dagger}_{\nu_1}|\mathrm{vac}\rangle$ and $\langle\nu_1,\cdots,\nu_n|\underset{\mathrm{def}}{=}\langle\mathrm{vac}|\psi_{\nu_1}\cdots\psi_{\nu_n}$, $n=0,1,2,\cdots$, and indeed these elements constitute mutually dual basis of both spaces: $\langle\mu_1,\cdots,\mu_m|\nu_n,\cdots,\nu_1\rangle=0$ if $m\neq n$, $=\det(\delta_{\mu_i\nu_j})$ if $m=n$.

Let g be an element of the Clifford group $G(W)$. The rotation in W induced by g, $T_g: w\mapsto gwg^{-1}$, is even or odd (i.e. $\det T_g=+1$ or -1) according as corank T_4=even or odd; in particular for a generic even/odd $g\in G(W)$ we have corank $T_4=0/1$ and expression (3)/(4) in [3] for $N(g)$. An element $w\in W$ itself belongs to $G(W)$ if and only if $\langle w,w\rangle\neq 0$, in which case we have $wg\in G(W)$. First consider an even generic g, so that we have, with the abbreviation $\langle g\rangle\underset{\mathrm{def}}{=}\langle\mathrm{vac}|g|\mathrm{vac}\rangle$,

$$N(g)=\langle g\rangle e^{L},\qquad L=\frac{1}{2}(\psi^{\dagger}\psi)\begin{pmatrix}S_1-1 & S_2\\ S_3 & S_4-1\end{pmatrix}\begin{pmatrix}{}^t\psi\\ -{}^t\psi^{\dagger}\end{pmatrix}\tag{21}$$

$${}^tS_1=S_4,\quad {}^tS_2=-S_2,\quad {}^tS_3=-S_3$$

where $S_g=\begin{pmatrix}S_1 & S_2\\ S_3 & S_4\end{pmatrix}$ is related to $T_g=\begin{pmatrix}T_1 & T_2\\ T_3 & T_4\end{pmatrix}$ through the reciprocal formulas

$$S_g=\begin{pmatrix}1 & -T_2\\ & 1\end{pmatrix}\begin{pmatrix}T_1 & \\ & T_4^{-1}\end{pmatrix}\begin{pmatrix}1 & \\ T_3 & 1\end{pmatrix},$$
(22)
$$T_g=\begin{pmatrix}1 & -S_2\\ & 1\end{pmatrix}\begin{pmatrix}S_1 & \\ & S_4^{-1}\end{pmatrix}\begin{pmatrix}1 & \\ S_3 & 1\end{pmatrix}.$$

Then we have, letting $w=(\psi^\dagger\psi)\begin{pmatrix}c^\dagger\\ c\end{pmatrix}$,

(23) $$N(wg)=\langle g\rangle w_1 e^L,\qquad w_1=(\psi^\dagger\psi)\begin{pmatrix}c^\dagger+S_2c\\ S_4c\end{pmatrix},$$

(24) $$N(gw)=\langle g\rangle w_2 e^L,\qquad w_2=(\psi^\dagger\psi)\begin{pmatrix}S_1c^\dagger\\ c+S_3c^\dagger\end{pmatrix}.$$

For an odd generic g' (so that $N(g')=w_0e^L$ with $w_0\in W$), the composition wg' or $g'w$ gives an even one, and

(25) $$N(wg')=\langle ww_0\rangle e^{L_1},\qquad L_1=L+\frac{1}{\langle ww_0\rangle}w_1\wedge w_0,$$

(26) $$N(g'w)=\langle w_0w\rangle e^{L_2},\qquad L_2=L+\frac{1}{\langle w_0w\rangle}w_0\wedge w_2,$$

where w_1 and w_2 are given by (23) and (24) respectively, using $S=S_g$, $N(g)=e^L$.

It should be noted also that T_g and $T_{g'}$ commute if and only if $g, g'\in G(W)$ either commute or anticommute.

Applying the above formulas to the case $w=\psi_\pm(x)$ and $L=L_F(a)$, we have, for w_1 in (23) and (25),

(27) $$w_1=\int_{-\infty}^{+\infty}\underline{du}\,\xi_\pm(x-a\,;u)e^{-im(a-u+a+u^{-1})}\psi(u),$$

where

$$\xi_\pm(x\,;u)=\sqrt{0+iu}^{\,\pm1}e^{-im(x-u+x+u^{-1})}+\int_0^\infty \underline{du'}\sqrt{0+iu'}^{\,\pm1}e^{-im(x-u'+x+u'^{-1})}\frac{i(u+u')}{u-u'-i0}.$$

Then $\xi=\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$ is analytically continued to the complex region of x such that $\operatorname{Im}x^\pm<0$, satisfies the Dirac equation $\partial_{x^\pm}\xi_\pm=\pm m\xi_\mp$ there, and shows a strict Fermi-type behavior at $x=0$ in the Euclidean region. Indeed we have

(28) $$\xi(x\,;u)=\frac{1}{2}(w_0(-x^-,x^+)+w_0^*(-x^-,x^+))+\textstyle\sum_{l=1}^\infty((iu)^l w_l(-x^-,x^+)+(iu)^{-l}w_l^*(-x^-,x^+)).$$

Combining (23)$\sim$(28) we obtain the following operator expansions for $\psi(x)\varphi_F(a)$ and $\psi(x)\varphi^F(a)$:

(29) $$N(\psi(x)\varphi_F(a))=\varphi_0^F(a)\frac{1}{2}(w_0[a]+w_0^*[a])+\textstyle\sum_{l=1}^\infty(\varphi_l^F(a)w_l[a]+\varphi_{-l}^F(a)w_l^*[a]),$$

(30) $$N(\psi(x)\varphi^F(a))=e^{L_F(a)}\frac{i}{2}(w_0[a]-w_0^*[a])$$

$$+\sum_{l=1}^{\infty}(\varphi_{F,l}(a)w_l[a]+\varphi_{F,-l}(a)w_l^*[a]),$$

where

$$\varphi_l^F(a)=\psi_l(a)e^{L_F(a)},\qquad \varphi_{F,l}(a)=\psi_l(a)\psi_0(a)e^{L_F(a)},$$

(31)
$$\psi_l(a)=\int_{-\infty}^{+\infty}\underline{du}(iu)^l e^{-im(a^- u+a^+u^{-1})}\psi(u)\qquad (l\in \boldsymbol{Z}).$$

Here $w_l[a]$ denotes $w_l(-x^-+a^-, x^+-a^+)$ and similarly for $w_l^*[a]$. Since the norm is linear,

$$N(d\varphi_F)=dN(\varphi_F)=dL_F\cdot e^{L_F}\quad\text{and}\quad N(d\varphi^F)=(d\psi_0+\psi_0 dL_F)e^{L_F}.$$

Noting the relations $dL_F(a)=(-i\psi_1(a)d(-a^-)+i\psi_{-1}(a)da^+)\psi_0(a)$ and $d\psi_l(a)=\psi_{l+1}(a)md(-a^-)+\psi_{l-1}(a)mda^+$, we obtain

(32) $$N(d\varphi_F(a))=-i\varphi_{F,1}(a)md(-a^-)+i\varphi_{F,-1}(a)mda^+,$$

(33) $$N(d\varphi^F(a))=\varphi_1^F(a)md(-a^-)+\varphi_{-1}^F(a)mda^+.$$

Finally we give the commutation relations satisfied by our field operators when placed in mutually space-like positions.

First, the above mentioned fact that g and $g'\in G(W)$ either commute or anti-commute if T_g and $T_{g'}$ commute, together with the Lorentz covariance of φ_F and φ^F, yields micro-causality for φ_F and φ^F:

(34) $$\varphi_F(x)\varphi_F(x')=\varphi_F(x')\varphi_F(x),\qquad \varphi^F(x)\varphi^F(x')=\varphi^F(x')\varphi^F(x),\quad\text{for } (x'-x)^2<0.$$

Of course, ψ satisfies

(35) $$\psi(x)\psi(x')=-\psi(x')\psi(x),\qquad \text{for } (x'-x)^2<0,$$

or more precisely

(36) $$\begin{pmatrix}[\psi_+(x),\psi_+(x')]_+ & [\psi_+(x),\psi_-(x')]_+\\ [\psi_-(x),\psi_+(x')]_+ & [\psi_-(x),\psi_-(x')]_+\end{pmatrix}=m^{-1}\begin{pmatrix}\partial_{x^-} & m\\ -m & \partial_{x^+}\end{pmatrix}\Delta(x-x';m^2)$$

where

$$\Delta(x;m^2)=i\int_{-\infty}^{\infty}\underline{du}\varepsilon(u)e^{-im(x^- u+x^+u^{-1})}=\begin{cases}\varepsilon(x^0)J_0(m\sqrt{x^2}) & x^2>0\\ 0 & x^2<0.\end{cases}$$

On the other hand, the definition (6) in [3] of φ_F reads: $T_{\varphi_F(x)}(\psi(x'))=\pm\psi(x')$ if $(x'-x)^2<0$ and $x'^1-x^1\lesseqgtr 0$ (i.e. if $x'^+\gtreqless x^+$ and $x'^-\lesseqgtr x^-$), while φ^F is defined by $T_{\varphi^F(x)}(\psi(x'))=\mp\psi(x')$ with the same x and x'. These definitions are readily rewritten as follows:

(37) $$\varphi_F(x)\psi(x')=\pm\psi(x')\varphi_F(x),\qquad \varphi^F(x)\psi(x')=\mp\psi(x')\varphi^F(x),\quad\text{for } x'^+\gtreqless x^+,\ x'^-\lesseqgtr x^-.$$

(34) and (37), when combined with (29) and (30), now yield

(38) $$\varphi_F(x)\varphi^F(x')=\pm\varphi^F(x')\varphi_F(x)\qquad\text{for } x'^+\gtreqless x^+,\ x'^-\lesseqgtr x^-.$$

2. We now proceed to construction of the wave functions of $W_{a_1,\cdots,a_n}^{\mathrm{strict}}$ in terms of our field operators φ_F,φ^F and ψ. Let $x_1,\cdots,x_k$, $a_1,\cdots,a_n$ be $k+n$ Minkowski 2-vectors in mutually space-like positions. We introduce the k-fold wave functions with n branch points, $w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k;a_1,\cdots,a_n)$, for any ordered subset $(\nu_1,\cdots,\nu_m)$ of indices $\{1,\cdots,n\}$, as follows. Namely, if $m=0$ we define

$$w_{F,n}(x_1,\cdots,x_k;a_1,\cdots,a_n)$$
$$=\langle\text{vac}|\,\psi(x_1)\cdots\psi(x_k)\varphi_F(a_1)\cdots\varphi_F(a_n)\,|\text{vac}\rangle$$

and in general, we define $\operatorname{sgn}\begin{pmatrix}\nu_1,\cdots,\nu_m\\ \nu'_1,\cdots,\nu'_m\end{pmatrix}w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k;a_1,\cdots,a_n)$ (where $\{\nu_1,\cdots,\nu_m\}=\{\nu'_1,\cdots,\nu'_m\}$ and $\nu'_1<\cdots<\nu'_m$) to be a similar expression as above, with $\varphi_F(a_\nu)$ within the bracket being replaced by $\varphi^F(a_\nu)$ for $\nu=\nu_1,\cdots,\nu_m$. If $k=0$, our $w_{F,n}^{\nu_1,\cdots,\nu_m}$ should also be denoted by $\tau_{F,n}^{\nu_1,\cdots,\nu_m}(a_1,\cdots,a_n)$, since for $m=0$ (resp. $m=n$) it reduces to the n-point τ-function of φ_F (resp. φ^F) discussed in [3]. We often drop parameters $a_1,\cdots,a_n$ and denote them by $w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k)$ and $\tau_{F,n}^{\nu_1,\cdots,\nu_m}$. Also we use

$$\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k)=w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k)/\tau_{F,n},$$

and

$$\hat{\tau}_{F,n}^{\nu_1,\cdots,\nu_m}=\tau_{F,n}^{\nu_1,\cdots,\nu_m}/\tau_{F,n}.$$

Note that all these quantities represent 0 if $k+m$ is odd.

From (29), (30) and (37) it follows that our wave functions admit the local expansion of the form (3) with $l_0=0$ at each of $a_1,\cdots,a_n$, i.e. of the following form in the style of (10):

(39) $\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m}(x)\sim\sum_{l=0}^{\infty}\boldsymbol{c}_l(\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m})w_l[A]+\sum_{l=0}^{\infty}\boldsymbol{c}_l^*(\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m})w_l^*[A],$

and that the coefficients $\boldsymbol{c}_l(\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m})$ in this expansion are expressed in terms of τ-functions. Namely assuming $\nu_1<\cdots<\nu_m$ and $(a_\nu-a_{\nu'})^+>0$ for $\nu>\nu'$, the μ-th component of $\boldsymbol{c}_0(\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m})$ is

$$(40)\qquad (-)^{\#(\{1,\cdots,\mu-1\}\cap\{\nu_1,\cdots,\nu_m\})}\begin{cases}(1/2)\hat{\tau}_{F,n}^{\nu_1,\cdots,\nu_k,\mu,\nu_{k+1},\cdots,\nu_m} & \text{if } \nu_k<\mu<\nu_{k+1},\\ (i/2)\hat{\tau}_{F,n}^{\nu_1,\cdots,\nu_{k-1},\nu_{k+1},\cdots,\nu_m} & \text{if } \nu_k=\mu,\end{cases}$$

while from (32) and (33)

$$(41)\qquad {}^t(\tau_{F,n}\boldsymbol{c}_1(\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m}))=2\begin{pmatrix}m^{-1}\partial_{(-a_1^-)} & & \\ & \ddots & \\ & & m^{-1}\partial_{(-a_n^-)}\end{pmatrix}{}^t(\tau_{F,n}\cdot\boldsymbol{c}_0(\hat{w}_{F,n}^{\nu_1,\cdots,\nu_m})).$$

We note that (35) together with positive-definiteness of the inner product in $W_{a_1,\cdots,a_n}^{\text{strict},\boldsymbol{R}}$ yields several inequalities among Euclidean τ-functions.

The analytic prolongability of the vacuum expectation $\langle\text{vac}|\cdots|\text{vac}\rangle$ (or of any matrix element) of product of field operators in their arguments is well-known. Indeed, consider $\langle\text{vac}|\,\psi(x)\varphi^F(a)\,|\text{vac}\rangle$ for example, and expand it into

$$\sum_{l=0}^{\infty}\frac{1}{l!}\int_0^\infty\cdots\int_0^\infty\underline{du}_1\cdots\underline{du}_l\,\langle\text{vac}|\,\psi(0)\,|u_l\cdots u_1\rangle$$
$$\times\langle u_1\cdots u_l|\,\varphi^F(0)\,|\text{vac}\rangle\,e^{-im((x^--a^-)U+(x^+-a^+)U')}$$

with $U=u_1+\cdots+u_l$ and $U'=u_1^{-1}+\cdots+u_l^{-1}$, and we shall see that this quantity is analytically prolonged to the complex region of x and a satisfying $\operatorname{Im}(x^\pm-a^\pm)<0$. (Note that no role is played by the accidental fact that $\langle\text{vac}|\,\psi(0)\,|u_l\cdots u_1\rangle=0$ for $l\neq1$.) The same reasoning yields that our wave function $w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k)$, as the vacuum expectation of the product $\psi(x_1)\cdots\psi(x_k)\varphi(a_1)\cdots\varphi(a_n)$, with φ standing either for φ_F or for φ^F, admits an analytic prolongation to the region $Y^{k+n,C}$ of complexified arguments $x_1,\cdots,x_k, a_1,\cdots,a_n$ defined as follows:

$$Y^{n,C}=\{(x_1,\cdots,x_n)\in(X^C)^n \mid \mathrm{Im}\, x_\nu^\pm<\mathrm{Im}\, x_{\nu'}^\pm \text{ for } \nu<\nu'\},$$

where $(X^C)^n$ stands for the Cartesian product of n copies of X^C, the complexified space-time. We also set $Y^{n,\mathrm{Euc}}=Y^{n,C}\cap(X^{\mathrm{Euc}})^n$. Note that they are convex cones in $(X^C)^n$ resp. in $(X^{\mathrm{Euc}})^n$, and hence simply connected. From the above reasoning we also see that for $a_1,\cdots,a_n$ fixed and $\mathrm{Im}\, x^\pm$ tending to $-\infty$, the wave function $w_{F,n}^{\nu_1,\cdots,\nu_m}(x)$ tends to 0 exponentially.

The commutation relation (37) between $\psi(x)$ and $\varphi(a)$ implies that, if $(x-a)^2=4(x^+-a^+)(x^--a^-)<0$,

$$\langle\mathrm{vac}|\cdots\psi(x)\varphi_F(a)\cdots|\mathrm{vac}\rangle=\varepsilon(x^+-a^+)\langle\mathrm{vac}|\cdots\varphi_F(a)\psi(x)\cdots|\mathrm{vac}\rangle$$

and

$$\langle\mathrm{vac}|\cdots\psi(x)\varphi^F(a)\cdots|\mathrm{vac}\rangle=\varepsilon(x^--a^-)\langle\mathrm{vac}|\cdots\varphi^F(a)\psi(x)\cdots|\mathrm{vac}\rangle.$$

Since

$$\langle\mathrm{vac}|\cdots\psi(x)\varphi(a)\cdots|\mathrm{vac}\rangle \quad\text{and}\quad \langle\mathrm{vac}|\cdots\varphi(a)\psi(x)\cdots|\mathrm{vac}\rangle$$

are already known to be analytically prolonged to $\mathrm{Im}\,(x^\pm-a^\pm)<0$ and to $\mathrm{Im}\,(x^\pm-a^\pm)>0$ respectively, the above equalities imply that our $w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k)$, when prolonged to $Y^{k+n,C}$ and then restricted to $Y^{k+n,\mathrm{Euc}}$, is analytically prolongable in both ways, but with opposite signs, around each $\{x_\kappa=a_\nu\}$.

The commutation relation (38) between φ_F and φ^F have exactly the same effect as above, while those within ψ's, φ_F's and φ^F's have even simpler consequences on the property of our wave functions: analytic prolongability with no discrepancy of sign around each $\{x_\kappa=x_{\kappa'}\}$ etc. Summing up, we conclude that Euclidean $w_{F,n}^{\nu_1,\cdots,\nu_m}(x_1,\cdots,x_k)$, originally defined in $Y^{k+n,\mathrm{Euc}}$, is analytically prolongable to a double-valued function (whose 2 values differring only in signs) on the whole $(X^{\mathrm{Euc}})^{k+n}$ with its singularities appearing only along $\{x_\kappa=x_{\kappa'}\}$, $\{a_\nu=a_{\nu'}\}$, and $\{x_\kappa=a_\nu\}$ with $\kappa,\kappa'=1,\cdots,k$ and $\nu,\nu'=1,\cdots,n$, where the last ones and part of the second correspond to branch points.

The (Euclidean) wave function $w_{F,n}^{\nu_1,\cdots,\nu_m}(x)$, with its parameters $a_1,\cdots,a_n$ being distinct and fixed in X^{Euc}, is now a double-valued analytic function in $X^{\mathrm{Euc}}-\{a_1,\cdots,a_n\}$. Notice that the local expansion formula (39) does also imply the double-valued nature of our wave function around each a_ν; in fact it implies an even stronger fact that $w_{F,n}^{\nu_1,\cdots,\nu_m}(x)$ is of strict Fermi-type at each a_ν. We already know that $w_{F,n}^{\nu_1,\cdots,\nu_m}(x)$ tends to 0 exponentially at infinity in X^{Euc}. We can show further, by employing (13) and (14) in [3], that $w_{F,n}^{\nu_1,\cdots,\nu_m}(x)$ is real. We now conclude that our $w_{F,n}^{\nu_1,\cdots,\nu_m}(x)$ belongs to $W_{a_1,\cdots,a_n}^{\mathrm{strict},\,R}$.

References

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[3] ——: Proc. Japan Acad., **53A**, 6–10 (1977).