34. A Note on the Large Sieve. II

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1977)

1. Let $\mathfrak{N}$ be a set of integers in an interval of length N. Also let $\mathfrak{P}$ be a set of prime numbers p to each of which Ω_p a set of residues (mod p) is associated. It is assumed that $|\Omega_p|$ the number of elements of Ω_p satisfies $0<|\Omega_p|<p$. Then the large sieve under the present consideration is the problem of estimating

$$S=|\{n \in \mathfrak{N};\ n(\bmod p) \notin \Omega_p \quad \text{for all } p \in \mathfrak{P}\}|.$$

According to the famous theorem of Montgomery [2] (with the latter refinement [3]) we have

$$(1) \qquad S \leqq (N+Q^2)\Big\{\sum_{q \in \mathcal{Q}} \prod_{p|q} \frac{|\Omega_p|}{p-|\Omega_p|}\Big\}^{-1}$$

where

$$\mathcal{Q}=\Big\{q \leqq Q;\ q \Big| \prod_{p \in \mathfrak{P}} p\Big\}.$$

Kobayashi [1] made an important observation that the optimal value of the Selberg λ_d (see (2) below) can be put into an expression which combines well with the dual form of the (additive) large sieve inequality, and thus he got a proof of (1) via Selberg's procedure.

The purpose of the present note is to show that there is a simpler modification of Selberg's argument than Kobayashi's which leads us to (1) quite straightforwardly. In particular we do not need the explicit value of λ_d. But as [1] we have to appeal to the following result due to Montgomery and Vaughan [3; formula (2.3)]:

Lemma. *Let $\{x_j\}$ be a set of real numbers which are δ well-spaced* (mod 1). *Then, for any complex numbers b_j and real M and $N(>0)$, we have*

$$\sum_{M<n\leqq M+N} \Big|\sum_j b_j e^{2\pi i x_j n}\Big|^2 \leqq (N+\delta^{-1}) \sum_j |b_j|^2.$$

2. In order to simplify the notations we introduce the following conventions that $\Omega_d=\Omega_{p_1}\times\Omega_{p_2}\times\cdots\times\Omega_{p_t}$ if $\mathcal{Q} \ni d=p_1p_2\cdots p_t$ and that $n \in \Omega_d$ means n (mod d) $\in \Omega_d$, so $n \in \Omega_1$ for any n.

Then by the fundamental idea of Selberg we have

$$(2) \qquad S \leqq \sum_{M<n\leqq M+N} \Big|\sum_{n \in \Omega_d} \lambda_d\Big|^2,$$

where $\mathfrak{N} \subseteqq (M, M+N]$ and λ_d are complex numbers defined on $\mathcal{Q}$ whose values are arbitrary, except for

(3) $$\lambda_1=1.$$

It is easy to see that the characteristic function of the set of integers n such that $n\in\Omega_d$ is given by

$$\frac{1}{d}\sum_{h=1}^{d}\sum_{l\in\Omega_d}\exp\left(2\pi i\frac{h}{d}(n-l)\right)=\frac{1}{d}\sum_{q|d}\sum_{\substack{r=1\\(q,r)=1}}^{q}\sum_{l\in\Omega_d}\exp\left(2\pi i\frac{r}{q}(n-l)\right).$$

But obviously

$$\sum_{l\in\Omega_d}\exp\left(-2\pi i\frac{r}{q}l\right)=\frac{|\Omega_d|}{|\Omega_q|}\sum_{l\in\Omega_q}\exp\left(-2\pi i\frac{r}{q}l\right).$$

Thus (2) can be written as

$$S\leqq\sum_{M<n\leqq M+N}\Bigg|\sum_{q\in Q}|\Omega_q|^{-1}$$
$$\times\sum_{\substack{r=1\\(q,r)=1}}^{q}\exp\left(2\pi i\frac{r}{q}n\right)\left(\sum_{d\equiv 0(\mathrm{mod}\, q)}\frac{\lambda_d}{d}|\Omega_d|\right)\left(\sum_{l\in\Omega_d}\exp\left(-2\pi i\frac{r}{q}l\right)\right)\Bigg|^2.$$

Hence, by the lemma and by that r/q in the above sum are Q^{-2} well-spaced (mod 1), we have

$$S\leqq(N+Q^2)\sum_{q\in Q}|\Omega_q|^{-2}\sum_{\substack{r=1\\(q,r)=1}}^{q}\left|\sum_{l\in\Omega_q}\exp\left(-2\pi i\frac{r}{q}l\right)\right|^2\left|\sum_{d\equiv 0(\mathrm{mod}\, q)}\frac{\lambda_d}{d}|\Omega_d|\right|^2.$$

Here we note

$$\sum_{\substack{r=1\\(q,r)=1}}^{q}\left|\sum_{l\in\Omega_q}\exp\left(-2\pi i\frac{r}{q}l\right)\right|^2$$
$$=\prod_{p|q}\left\{\sum_{u=1}^{p-1}\left|\sum_{l\in\Omega_p}\exp\left(-2\pi i\frac{u}{p}l\right)\right|^2\right\}=\prod_{p|q}|\Omega_p|(p-|\Omega_p|).$$

So we find

(4) $$S\leqq(N+Q^2)\sum_{q\in Q}\prod_{p|q}(p|\Omega_p|^{-1}-1)\left|\sum_{d\equiv 0(\mathrm{mod}\, q)}\frac{\lambda_d}{d}|\Omega_d|\right|^2.$$

Now we put

$$y_q=\sum_{d\equiv 0(\mathrm{mod}\, q)}\frac{\lambda_d}{d}|\Omega_d|,$$

then the condition (3) is transformed into

$$\sum_{q\in Q}\mu(q)y_q=\lambda_1=1,$$

where $\mu(q)$ is the Moebius function. And by Schwarz's inequality we get, for certain optimal y_q,

$$\left\{\sum_{q\in Q}|y_q|^2\prod_{p|q}(p|\Omega_p|^{-1}-1)\right\}\left\{\sum_{q\in Q}\prod_{p|q}\frac{|\Omega_p|}{p-|\Omega_p|}\right\}=1.$$

Combined with (4), this gives rise to (1).

3. By the way we remark that a large sieve extension of a recent sieve result of Selberg [4] can be expressed in the following form:

$$\sum_{\substack{qr\leqq Q\\(q,r)=1\\r\in Q}}\frac{q}{\varphi(q)\prod_{p|r}|\Omega_p|(p-|\Omega_p|)}\sum_{\chi(\mathrm{mod}\, q)}^{*}\left|\sum_{M<n\leqq M+N}\chi(n)\Psi_r(n,\Omega)a_n\right|^2$$
$$\leqq(N+Q^2)\sum_{M<n\leqq M+N}|a_n|^2,$$

where a_n are arbitrary complex numbers and

$$\Psi_r(n, \Omega)=\prod_{p|r}\left(\sum_{l\in\Omega_p} c_p(n-l)\right)$$

with the Ramanujan sum $c_p(n-l)$.

References

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[4] A. Selberg: Remarks on sieves. Proc. 1972 Number Theory Conf. Boulder, 205–216.