33. A Counterexample to a Conjecture By P. Erdös

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1. Ch. Pommerenke [4] proved the following theorem. *Let* $f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$ *be a polynomial of degree* n *with some* $a_j \neq 0$. *Assume that the region* $E_f = \{z \in \boldsymbol{C} : |f(z)| \leqq 1\}$ *is connected, where* $\boldsymbol{C}$ *stands for the field of complex numbers. Then*

$$\max_{z \in E_f} |f'(z)| < \frac{en^2}{2}.$$

P. Erdös [5] reviewing Pommerenke's paper conjectured that

$$\max_{z \in E_f} |f'(z)| < \frac{n^2}{2}$$

is also true and it is best possible. Erdös reposed his conjecture as a problem in [2]. As it appears in [3] Erdös' conjecture was unsolved until the year 1972 and to the best of our knowledge it is open until now. The purpose of this paper is to give a counterexample to Erdös' conjecture. It seems to us that this gives some information concerning the famous coefficient conjecture of L. Bieberbach [1], [6], [7].

2. Counterexample to Erdös' conjecture. Let $T_n(z)$ be the Chebyshev polynomial of degree n, defined by $T_n(z) = 2\cos n\theta$, where $z = 2\cos\theta$, and $n = 0, 1, 2, 3, \cdots$. This is a complex polynomial of a real variable and has n real zeros in the line segment $[-2, 2]$ and $-2 \leq T_n(z) \leq 2$ for $-2 \leq z \leq 2$. The recursion formula, $T_{n+1}(z) = zT_n(z) - T_{n-1}(z)$, which is valid since $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta \cos\theta$, allows us to write the following sequence of polynomials: $T_0(z) = 2$, $T_1(z) = z$, $T_2(z) = z^2 - 2$, $T_3(z) = z^3 - 3z$, $T_4(z) = z^4 - 4z^2 + 2$ and in general

$$T_n(z) = z^n + \sum_{m=1}^{[n/2]} (-1)^m \frac{n}{m} \binom{n-m-1}{m-1} z^{n-2m}$$

is a complex inhomogeneous polynomial in a real variable and of degree n. Consider now $f(z) = \lambda^n T_n(z/\lambda)$. This is a monic inhomogeneous polynomial of degree n and in fact $-2\lambda^n \leq f(z) \leq 2\lambda^n$ for $-2\lambda \leq z \leq 2\lambda$. Take $\lambda = 1/2^{1/n}$. Then $-1 \leq f(z) \leq 1$ for $-2/2^{1/n} \leq z \leq 2/2^{1/n}$. Because of the fact that $T_n(z) = T_n(2\cos\theta) = 2\cos n\theta$, it implies that $T'_n(2\cos\theta) = n(\sin n\theta / \sin\theta)$. Thus, $\max\{|T'_n(z)| : -2/2^{1/n} \leq z \leq 2/2^{1/n}\} = n^2$ because $\max\{(\sin n\theta/\sin\theta) : -2/2^{1/2} \leq z \leq 2/2^{1/n}\} = n$. However, $f(z) = \lambda^n T_n(z/\lambda)$. Therefore $f'(z) = \lambda^{n-1} T'_n(z/\lambda)$ and so $\max\{|f'(z)| : -2\lambda \leq z \leq 2\lambda\} = \lambda^{n-1} n^2$. If we set $\lambda = 1/2^{1/n}$, then $\max\{|f'(z)| : -2/2^{1/n} \leq z \leq 2/2^{1/n}\}$

$=n^2/2\cdot 2^{1/n}>n^2/2$.

Claim that $E_f=\{z\in C:|f(z)|\leq 1\}$ is a connected subset of C. Assume that this is not the case. Then $E_f=A\cup B$ where A, B are disjoint, closed and nonempty subsets of C. It follows that $|f(z)|=1$ when $z\in\partial A$ (the topological boundary of A) by the analyticity of f. Thus if f has no zeros in A then the minimum modulus principle implies that $|f(z)|=1$ in A and which implies that $f(z)=$constant on C, which is a contradiction. Hence, f has a zero $x_1\in A$ and in fact this is a real zero. The same reasoning shows that f has a real zero, x_2 in B. Then the closed line segment $[x_1, x_2]$ with end points x_1, x_2 is contained in $E_f=A\cup B$, since $|f(z)|\leq 1$ on the closed real line segment between any two zeros of f which again is a contradiction, for the closed line segment $[x_1, x_2]$ is connected and $x_1\in A$, $x_2\in B$ where A, B are disjoint and closed sets in C. Thus E_f is connected. Hence we have given an inhomogeneous polynomial $f(z)$ of degree n with E_f connected subset of C but $\max_{z\in E_f}|f'(z)|>n^2/2$. □

3. **Remark.** For a better understanding of the set E_f we construct the following figures, as the degree n of the polynomial $f(z)$ varies. Let $n=2$. Then $T_2(z)=z^2-2$, $f(z)=z^2-1$. Consider $u(z)=\log|z-1|+\log|z+1|$. Then $u(z)$ is a harmonic function on $C-\{-1,1\}$. It follows that $u(z)=0$ on the lemniscate and $u(z)=\infty$ as $|z|=\infty$. Therefore $u(z)>0$ outside the lemiscate. It is clear that $u(z)<0$ inside the lemniscate. The picture of E_f is the shadowed region in Fig. 1, and $\{z\in C:|f(z)|=1\}=\{-2\lambda, 0, 2\lambda\}$.

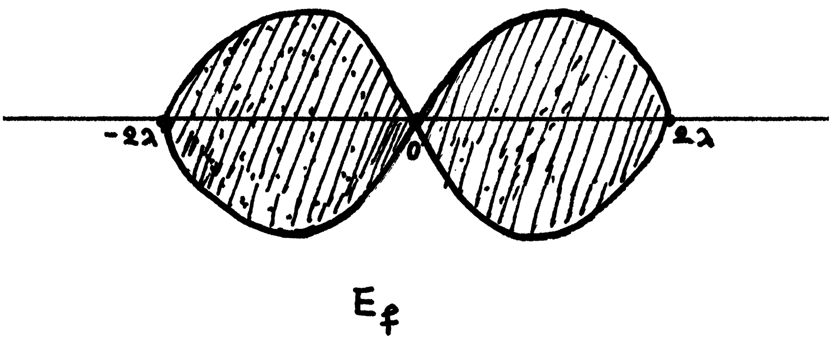

Fig. 1

Similarly, working for $n=3$ we find for E_f the shadowed region given by Fig. 2, and for $n=4$, we find for E_f the shadowed region given by Fig. 3. In a similar manner we obtain the figures for E_f, as $n\geq 5$.

4. **Open problem.** Find the least upper bound of the $\max_{z\in E_f}|f'(z)|$?

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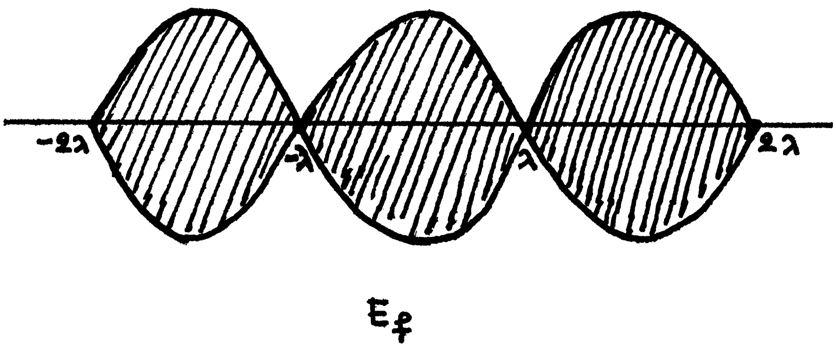

Fig. 2

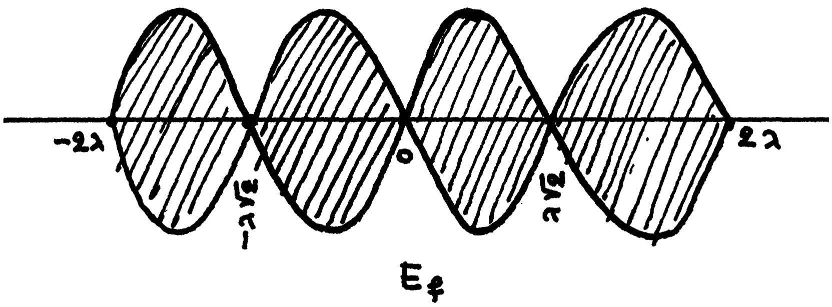

Fig. 3

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