31. A Note on Steenrod Operations in the Eilenberg-Moore Spectral Sequence

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1. Introduction and results. Let X be an associative H-space and BX the classifying space of X. The purpose of this note is to describe two kinds of Steenrod operations in the Eilenberg-Moore spectral sequence $\{E_r\}$ such that

$$E_2 \cong \mathrm{Cotor}_{H^*(X;\,Z_p)}(Z_p, Z_p) \Rightarrow H^*(BX; Z_p),$$

where p is a prime.

Our results are stated as follows.

Theorem 1. *In the Eilenberg-Moore spectral sequence $\{E_r\}$ there are Steenrod operations*

$$\beta^\varepsilon \mathcal{P}^i : E_r^{s,t} \to E_r^{s,t+2i(p-1)+\varepsilon}, \qquad 2i \leq t,$$

and

$$\beta^\varepsilon \mathcal{P}^i : E_r^{s,t} \to E_r^{s+(2i-t)(p-t)+\varepsilon,pt}, \qquad 2i \geq t,$$

where $r \geq 2$ and $\varepsilon = 0$ or 1.

Remark. If $p=2$, we understand $\mathcal{P}^i = Sq^{2i}$ and $\beta\mathcal{P}^i = Sq^{2i+1}$.

Theorem 2. *Let $u \in E_r^{s,t}$.*

(i) *If $2i \leq t-r+1$, then $d_r\beta^\varepsilon\mathcal{P}^i u = (-1)^\varepsilon \beta^\varepsilon \mathcal{P}^i d_r u$.*

(ii) *If $t-r+1 \leq 2i \leq t$, then $\beta^\varepsilon\mathcal{P}^i u$ survives to $E_q^{s,t+2i(p-1)+\varepsilon}$, where $q = r+(2i-t+r-1)(p-1)+\varepsilon$, $\beta^\varepsilon\mathcal{P}^i d_r u$ survives to $E_q^{s+q,t+2i(p-1)+\varepsilon+q-1}$, and*

$$d_q\beta^\varepsilon\mathcal{P}^i u = (-1)^\varepsilon\beta^\varepsilon\mathcal{P}^i d_r u.$$

(iii) *If $2i \geq t$, then $\beta^\varepsilon\mathcal{P}^i u$ survives to $E_q^{s+(2i-t)(p-1)+\varepsilon,pt}$, where $q = rp-p+1$, $\beta^\varepsilon\mathcal{P}^i d_r u$ survives to $E_q^{s+(2i-t)(p-1)+\varepsilon+q,pt+q-1}$, and*

$$d_q\beta^\varepsilon\mathcal{P}^i u = (-1)^\varepsilon\beta^\varepsilon\mathcal{P}^i d_r u.$$

Theorem 3. *Let $p: F^{s,t} = F^{s,t}H^{s+t}(BX; Z_p) \to E_\infty^{s,t}$ be the natural projection and $u \in F^{s,t}$.*

(i) *If $2i \leq t$, then $\beta^\varepsilon\mathcal{P}^i u \in F^{s,t}$ and $p\beta^\varepsilon\mathcal{P}^i u = \beta^\varepsilon\mathcal{P}^i pu$.*

(ii) *If $2i \geq t$, then $\beta^\varepsilon\mathcal{P}^i u \in F^{s+(2i-t)(p-1)+\varepsilon,t-(2i-t)(p-1)-\varepsilon}$ and $p\beta^\varepsilon\mathcal{P}^i u = \beta^\varepsilon\mathcal{P}^i pu$.*

Let $A = H^*(X; Z_p)$. It is well known that two kinds of Steenrod operations are defined on $\mathrm{Cotor}_A(Z_p, Z_p)$, that is, the vertical Steenrod operations

$$\beta^\varepsilon\mathcal{P}_V^i : \mathrm{Cotor}_A^{s,t} \to \mathrm{Cotor}_A^{s,t+2i(p-1)+\varepsilon}, \qquad 2i \leq t,$$

and the diagonal Steenrod operations

$$\beta^\varepsilon\mathcal{P}_D^i : \mathrm{Cotor}_A^{s,t} \to \mathrm{Cotor}_A^{s+(2i-t)(p-1)+\varepsilon,pt}, \qquad 2i \geq t,$$

which satisfy the usual properties such as Cartan formula and Adem relations. The vertical operations are induced by the topological Steenrod operations on A and the diagonal operations are algebraically defined on the cohomology of the Hopf algebra A.

Theorem 4. (i) *If* $2i \leq t$, *then* $\beta^{\varepsilon}\mathcal{P}^i : E_2^{s,t} \to E_2^{s,t+2i(p-1)+\varepsilon}$ *coinsides with* $\beta^{\varepsilon}\mathcal{P}_V^i$.

(ii) *If* $2i \geq t$, *then* $\beta^{\varepsilon}\mathcal{P}^i : E_2^{s,t} \to E_2^{s+(2i-t)(p-1)+\varepsilon,pt}$ *coinsides with* $\beta^{\varepsilon}\mathcal{P}_D^i$.

We write $\overline{\mathcal{P}}^i$ for $\mathcal{P}^{i+t} : E_r^{s,2t} \to E_r^{s+2i(p-1),2pt}$ if $p \geq 3$ and $\overline{Sq}^i$ for Sq^{i+t}: $E_r^{s,t} \to E_r^{s+i,2t}$ if $p=2$.

Theorem 5. (i) *Let* $p \geq 3$ *and* $u \in E_r^{s,2t}$. *Then*

$$\mathcal{P}^{pa}\overline{\mathcal{P}}^b u = \overline{\mathcal{P}}^b \mathcal{P}^a u,$$
$$\mathcal{P}^{pa+i}\overline{\mathcal{P}}^b u = 0 \qquad 0 < i < p.$$

(ii) *Let* $p=2$ *and* $u \in E_r^{s,t}$. *Then*

$$Sq^{2a}\overline{Sq}^b u = \overline{Sq}^b Sq^a u,$$
$$Sq^{2a+1}\overline{Sq}^b u = 0.$$

Remark. If $p=2$, our results are contained in Singer [7].

2. Outline of proofs. Let π be the cyclic group of order p. Let W be the standard $Z_p\pi$-free resolution with $W_0 = Z_p\pi e_0$.

Let $S_*(Y)$ be the singular chain complex of a space Y with all vertices at the base point. We define

$$K_{i,*} = S_*(EX) \otimes S_*(X) \otimes \cdots (i\text{-times}) \cdots \otimes S_*(X).$$

Then K forms a bisimplicial Z_p-double complex and $T(K)$ the total Z_p-complex.

Lemma 1. *There is a natural map of* Z_p*-complexes*

$$\varphi : W \otimes T(K) \to T(K)^p$$

such that (1) φ *is* π*-equivariant*, (2) $\varphi(w \otimes k) = k^p$, *where* k *is a* 0*-simplex and* $w \in W$, (3) $\varphi(e_0 \otimes k) = e_0 \otimes \xi(k^p)$, *where* $k \in T(K)$ *and* ξ *is the Alexander-Whitney map and*

$$\varphi(W \otimes T_j(K)) \subset \sum_{k \leq pj} [T(K)^p]_k.$$

Proof. The map φ is componentwise defined as follws.

$$\begin{array}{l} W_k \otimes C_{s,t}(K) \\ \quad \big\downarrow D\otimes D \\ \displaystyle\sum_{i+j=k} W_i \otimes W_j \otimes C_{s,t}(K^p) \\ \quad \big\downarrow \varphi^v \\ \displaystyle\sum_{i+j=k} W_i \otimes \sum_{t_1+\cdots+t_p=t+j} C_{s,t_1}(K) \otimes \cdots \otimes C_{s,t_p}(K) \\ \quad \big\downarrow \varphi^h \\ \displaystyle\sum_{i+j=k} \sum_{\substack{t_1+\cdots+t_r=t+j \\ s_1+\cdots+s_r=s+i}} C_{s_1,t_1}(K) \otimes \cdots \otimes C_{s_p,t_p}(K). \end{array}$$

Here D is the diagonal map, and φ^v (resp. φ^h) is constructed with respect to the vertical (resp. horizontal) degree by using the method of Dold [1]. Q.E.D.

Thus we can define Steenrod operations in $H^*(T(K))$ in a usual manner and we have the following lemmas.

Lemma 2. *The isomorphism*

$$H^*(BX\,;Z_p)\to H(T(K))=\mathrm{Cotor}_{S^*(X)}(S^*(EX),Z_p)$$

preserves Steenrod operations.

Lemma 3.
$$\varphi(W_k\otimes F_rT(K))\subset F_{r+k}T(K)^p,$$
$$\varphi(W_k\otimes F_rT(K))\subset F_{pr}T(K)^p.$$

This enables us to introduce Steenrod operations in the Eilenberg-Moore spectral sequence and the routine argument (see, for example, May [4]) shows our results.

3. Some applications. The results can be used to show the collapsing of the spectral sequence and to reproduce the data lost in passing to quotient and hence to obtain the cohomology $H^*(BX\,;Z_p)$.

Let G_2, F_4, E_6, E_7, E_8 be the compact, simply connected simple exceptional Lie groups of the indicated ranks.

The collapsing of the Eilenberg-Moore spectral sequence is not algebraically trivial when $H^*(X\,;Z_p)$ has p-torsion. However, we can show it by purely algebraic argument when (X,p) is given as follows: $(G_2,2),(F_4,2),(E_6,2),(E_7,2),(F_4,3),(E_6,3)$. Along this line we refer to Mimura-Mori [5]. The reader will compare with the works of Kono-Mimura [2], Kono-Mimura-Shimada [3] and Toda [8].

References

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