29. Fundamental Solutions to the Cauchy Problem of Some Weakly Hyperbolic Equation

By Atsushi YOSHIKAWA
Department of Mathematics, Hokkaido University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1977)

1. Consider the operator

$$L=D_t^2-t^{2m}\sum_{j,k=1}^n a_{jk}D_jD_k+b_0D_t+\sum_{j=1}^n b_jD_j+c.$$

Here m is a positive integer, and $a_{jk}=a_{jk}(t,x)$, $b_l=b_l(t,x)$, $c=c(t,x)C^\infty$ functions of $(t,x)=(t,x_1,\cdots,x_n)\in \boldsymbol{R}\times\boldsymbol{R}^n$. $D_t=-i\partial/\partial t$, $D_j=-i\partial/\partial x_j$, $j=1,\cdots,n$, and $i^2=-1$ as usual. We assume that $(a_{jk}(t,x))$ be a real symmetric positive definite matrix, reducing to the unit matrix for t,x sufficiently large.

2. Let $\tau\in\boldsymbol{R}$. Consider the following Cauchy problem:

$$(*)\qquad \begin{cases} Lv(t,x)=0, & t>\tau,\ x\in\boldsymbol{R}^n,\\ v(t,x)|_{t=\tau}=f_0(x), & D_tv(t,x)|_{t=\tau}=f_1(x),\end{cases}$$

f_0,f_1 being given distributions in $\mathcal{E}'(\boldsymbol{R}^n)$.

Let $\Delta=\{(t,\tau);\tau\leqq t\}$.

Definition. Let $U_j(t,\tau)$, $j=0,1$, be operators from $\mathcal{E}'(\boldsymbol{R}^n)$ to $\mathcal{D}'(\boldsymbol{R}^n)$ with kernels in $C^\infty(\Delta;\mathcal{D}'(\boldsymbol{R}^n\times\boldsymbol{R}^n))$. We call $U_j(t,\tau)$, $j=0,1$, a pair of fundamental solutions to the problem $(*)$ if

$$LU_j(t,\tau)=0,\ j=0,1,\qquad \text{in } \Delta,$$
$$D_t^kU_j(t,\tau)|_{t=\tau}=\delta_{jk}I,\qquad j,k=0,1,$$

δ_{jk} being the Kronecker symbol and I the identity operator.

3. The purpose of the present note is to construct a pair of fundamental solutions to the problem $(*)$ under the conditions explained below. We set

$$a(t,x,\xi)=\Big(\sum_{j,k=1}^n a_{jk}(t,x)\xi_j\xi_k\Big)^{1/2},\qquad \xi\in\boldsymbol{R}^n\backslash 0,$$

so that the principal symbol of L is

$$L_0(t,x,\xi_0,\xi)=(\xi_0-t^ma(t,x,\xi))(\xi_0+t^ma(t,x,\xi)).$$

We denote by $S_L(t,x,\xi_0,\xi)$ the subprincipal symbol of L. Thus,

$$S_L(t,x,\xi_0,\xi)=b_0(t,x)\xi_0+\sum_{j=1}^n b_j(t,x)\xi_j +it^{2m}\sum_{j,k=1}^n \xi_k\partial a_{jk}(t,x)/\partial x_j.$$

4. Set

$$C_{L\pm}(t,x,\xi)=S_L(t,x,\pm t^ma(t,x,\xi),\xi).$$

We assume

$$(1)\qquad C_{L\pm}(t,x,\xi)=t^{m-1}b(x,\xi)+t^mb_\pm(t,x,\xi).$$

Here $b(x,\xi)$ and $b_\pm(t,x,\xi)$ are smooth functions of t,x,ξ. For simplicity, we require that $\operatorname{Im}\{b(x,\xi)/|\xi|\}$ be uniformly bounded on $\boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$.

5. **Theorem.** *Under the assumption* (1), *there exists a unique*

pair of fundamental solutions to the problem $(*)$.

The requirement (1) is a variant of Levi's condition. This is imposed in the discussions of Oleinik [6]. See also Ohya [5].

6. Remark. Let $f \in C_0^\infty(\boldsymbol{R}^{n+1})$ and set

$$u(t,x)=i\int_{-\infty}^{t}[U_1(t,\tau)f(\tau,\cdot)](x)d\tau.$$

Then $Lu=f$ and $\inf\{t;(t,x)\in \operatorname{supp} u \text{ for some } x\}=\inf\{t;(t,x)\in \operatorname{supp} f$ for some $x\}$. That is,

$$E(t,\tau)=\begin{cases} iU_1(t,\tau), & t>\tau,\\ 0, & t\leqq\tau,\end{cases}$$

is a forward fundamental solution for the operator L (cf. Hörmander [4]). The assumption (1) is known to be necessary for the existence of a forward fundamental solution of the operator L (Ivrii-Petkov [3]).

7. The rest of the present note is devoted to a (sketchy) proof of Theorem. This is done via a "good" parametrix to the problem $(*)$. Let $\Delta^+=\{(t,\tau);0\leqq\tau\leqq t\}$.

Definition. Let $E_j(t,\tau)$, $0\leqq\tau\leqq t$, $j=0,1$, be operators from $\mathcal{E}'(\boldsymbol{R}^n)$ to $\mathcal{D}'(\boldsymbol{R}^n)$ with kernels in $C^\infty(\Delta^+;\mathcal{D}'(\boldsymbol{R}^n\times\boldsymbol{R}^n))$. We say that $E_j(t,\tau)$, $j=0,1$, form a good parametrix to the problem $(*)$ for $0\leqq\tau\leqq t$ if they satisfy

$$LE_j(t,\tau)=K_j(t,\tau),\qquad j=0,1,\text{ in }\Delta^+,$$
$$D_t^kE_j(t,\tau)|_{t=\tau}-\delta_{jk}I=R_{kj}(\tau),\qquad j,k=0,1,\ \tau\geqq 0.$$

Here $K_j(t,\tau)$, $j=0,1$, are integral operators with kernels in $C^\infty(\Delta^+\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ and $R_{jk}(\tau)$, $j,k=0,1$, with kernels in $C^\infty(\bar{\boldsymbol{R}}_+\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$.

8. For the construction of a good parametrix, we need the following symbol classes (cf. [7], [8]). Let κ be any positive integer.

Definition. For real μ,ν,λ, we denote by $S_{(\kappa)}^{\mu,\nu,\lambda}$ (resp. $S_{(\kappa)+}^{\mu,\lambda}$) the space of all C^∞ functions $p(t,\tau,x,\xi)$ on $\bar{\boldsymbol{R}}_+\times\bar{\boldsymbol{R}}_+\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ such that for any non-negative integers, k,l, and multi-indices α,β, we have

$$|D_t^kD_\tau^lD_x^\alpha D_\xi^\beta p(t,\tau,x,\xi)|$$
$$\leqq C(1+|\xi|)^{\mu-|\beta|}(|\xi|^{-1}+t^\kappa)^{(\nu-k)/\kappa}(|\xi|^{-1}+\tau^\kappa)^{(\lambda-l)/\kappa}$$
$$(\text{resp.}\leqq C(1+|\xi|)^{\mu-|\beta|}(|\xi|^{-1}+\tau^\kappa)^{(\lambda-l)/\kappa})$$

for $0\leqq t\leqq T_1$, $0\leqq\tau\leqq T_2$, $x\in K$. Here T_1,T_2 are any positive numbers, K any compact subset of $\boldsymbol{R}^n$, C a positive constant depending on $T_1,T_2,K,\alpha,\beta,k,l$.

Definition. For real μ, we denote by S_∞^μ the space of all C^∞ functions $p(t,\tau,x,\xi)$ on $\Delta^+\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ such that for any non-negative integers N,k,l, and multi-indices α,β,

$$|D_t^kD_\tau^lD_x^\alpha D_\xi^\beta p(t,\tau,x,\xi)|\leqq C\tau^N(1+|\xi|)^{\mu-|\beta|}$$

for all $0\leqq\tau\leqq t\leqq T$, $x\in K$, $|\xi|\geqq 1$. Here T is any positive number, K any compact subset of $\boldsymbol{R}^n$, and C a positive constant depending on N,T,K,k,l,α,β.

9. Let $\phi^\sigma(t,\tau,x,\xi)$, $\sigma^2=1$, be respectively solutions of

$$\phi_t^\sigma=\sigma t^m a(t, x, \phi_x^\sigma), \qquad \sigma^2 = 1,$$

with the initial condition $\phi^\sigma|_{t=\tau}=\langle x, \xi\rangle$ $(\tau\geqq 0)$. We may assume that ϕ^σ, $\sigma^2=1$, are well-defined in the large. We now set

$$M(\sigma)=-\frac{m}{2}+\frac{1}{2}\sup\{\sigma \operatorname{Im} b(x, \xi)/a(0, x, \xi)\}, \qquad \sigma^2=1,$$

the superimum being taken over $(x, \xi)\in \boldsymbol{R}^n\times(\boldsymbol{R}^n\backslash 0)$.

10. Proposition. *Under the assumption* (1), *there exists a good parametrix to the problem* $(*)$ *for* $0\leqq\tau\leqq t$. *More precisely, there are symbols*

$$p_{j\sigma}^0(t, \tau, x, \xi)\in S_{(m+1)}^{\varepsilon-j, M(\sigma)+\varepsilon, M(-\sigma)+(1-j)m+\varepsilon},$$
$$p_{j\sigma}^1(t, \tau, x, \xi)\in S_{(m+1)}^{\varepsilon-j, M(-\sigma)+(1-j)m+\varepsilon},$$
$$p_{j\sigma}^2(t, \tau, x, \xi)\in S_\infty^{\varepsilon-j}, \qquad \sigma^2=1,\ j=0, 1,$$

such that, for $P_{j\sigma}(t, \tau, x, \xi)=\sum_{k=0}^2 p_{j\sigma}^k(t, \tau, x, \xi)$,

$$[E_j(t, \tau)f_j](x) = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{\phi^\sigma(t, \tau, x, \xi)-\langle y, \xi\rangle\}}P_{j\sigma}(t, \tau, x, \xi)f_j(y)dyd\xi, \tag{2}$$

$j=0, 1$, *form a good parametrix for the problem* $(*)$ *for* $0\leqq\tau\leqq t$. *Here the integrals* (2) *are oscillatory ones over* $\boldsymbol{R}^n\times\boldsymbol{R}^n$. ε *is an arbitrary positive number and may be omitted when* $n=1$ *and* $b(x, \xi)/a(0, x, \xi)$ *is independent of* x.

11. We have shown the above Proposition for the case $m=1$ in [7], [8]. A close discussion has also been done in Alinhac [1]. The proof for general m goes in an analogous way to the case $m=1$. That is, the essential point rests on the asymptotic behaviors of confluent hypergeometric functions. In fact, the exponent $M(\sigma)$ appears in this way.

12. In view of (2), we may assume $E_0(t, \tau)$, $E_1(t, \tau)$ properly supported, by an obvious modification if necessary. Since L is a differential operator, $K_j(t, \tau)$ and $R_{jk}(\tau)$ are then automatically properly supported. We first construct a pair of fundamental solutions to the problem $(*)$ when $0\leqq\tau\leqq t$. This can be done in a similar way to Chazarain [2]. Since $R_{jk}(\tau)=D_t^k E_j(t, \tau)|_{t=\tau}-\delta_{jk}I$, $j, k=0, 1$, are smoothing, $E_j'(t, \tau)=E_j(t, \tau)-R_{j0}(\tau)-i(t-\tau)R_{j1}(\tau)$, $j=0, 1$, also form a good parametrix, satisfying the initial conditions now exactly, and $K_j'(t, \tau)=LE_j'(t, \tau)$ have properly supported C^∞ kernels in $\Delta^+\times\boldsymbol{R}^n\times\boldsymbol{R}^n$. Let

$$[G(t, \tau)h](x) = i\int_\tau^t [E_1'(t, s)h(s, \tau, \cdot)](x)ds$$

for $h(s, \tau, \cdot)\in C^\infty(\Delta^+\times\boldsymbol{R}^n)$. Then $D_t^k G(t, \tau)|_{t=\tau}=0$, $k=0, 1$, and $LG(t, \tau)h=h+R(t, \tau)h$, where

$$[R(t, \tau)h](x)=i\int_\tau^t [K_1'(t, s)h(s, \tau, \cdot)](x)ds.$$

Let $B_q=\{x\in\boldsymbol{R}^n; |x|\leqq q\}$, q any positive integer, and $\chi_q(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_q=1$ on B_q, $\operatorname{supp}\chi_q\subset B_{q+1}$. Let $R_q(t, \tau)=\chi_q R(t, \tau)\chi_q$. Then since

R is properly supported, we have, for each $h \in C^\infty(\Delta^+; C_0^\infty(\boldsymbol{R}^n))$, $[I+R]h = [I+R_q]h$ for sufficiently large q. By solving the Volterra integral equation, we see $I+R_q$ invertible in each $C^0([0,T]; C^0(B_{q+1}))$. It then follows immediately that $I+R$ is invertible in $C^\infty(\Delta^+; C_0^\infty(\boldsymbol{R}^n))$ and so in $C^\infty(\Delta^+; C^\infty(\boldsymbol{R}^n))$. Let $G'(t,\tau) = G(t,\tau)(I+R(t,\tau))^{-1}$ and set

$$U_j^+(t,\tau) = E_j'(t,\tau) - G'(t,\tau)K_j'(t,\tau), \qquad j=0,1.$$

Then $U_j^+(t,\tau)$, $j=0,1$, are a pair of fundamental solutions to the problem $(*)$ for $t \geqq \tau \geqq 0$. In particular, for each t,τ, $U_j^+(t,\tau)$ map $\mathcal{E}(\boldsymbol{R}^n)$ into $\mathcal{E}(\boldsymbol{R}^n)$ and $\mathcal{E}'(\boldsymbol{R}^n)$ into $\mathcal{E}'(\boldsymbol{R})$.

13. Note that the same construction is also valid for the problem $(*)$ when $t \leqq s \leqq 0$, $(\tau = s)$, by changing t to $-t$. We thus obtain a pair of fundamental solutions $U_j^-(t,s)$, $t \leqq s \leqq 0$, $j=0,1$. Let us set

$$\Phi(t,s) = \begin{pmatrix} U_0^-(t,s) & U_1^-(t,s) \\ D_tU_0^-(t,s) & D_tU_1^-(t,s) \end{pmatrix} \qquad \text{for } t \leqq s \leqq 0.$$

This defines a mapping $\mathcal{E}(\boldsymbol{R}^n) \times \mathcal{E}(\boldsymbol{R}^n) \to \mathcal{E}(\boldsymbol{R}^n) \times \mathcal{E}(\boldsymbol{R}^n)$ and $\mathcal{E}'(\boldsymbol{R}^n) \times \mathcal{E}'(\boldsymbol{R}^n) \to \mathcal{E}'(\boldsymbol{R}^n) \times \mathcal{E}'(\boldsymbol{R}^n)$ for each $t \leqq s \leqq 0$.

Lemma. *There is a mapping* $\Psi(t,s)$, $t \leqq s \leqq 0$, *such that* $\Psi(t,s)\Phi(t,s) = I$.

14. Remark. Since L is strictly hyperbolic in $t<0$, we see immediately $\Psi(t,s) = \Phi(t,s)^{-1} = \Phi(s,t)$ if $s<0$, $\Phi(s,t)$ being essentially the evolution operator for $t,s<0$.

15. Proof of Lemma. Let L^* be the formal adjoint of L. Then since $S_{L^*}(t,x,\xi_0,\xi) = \overline{S_L(t,x,\xi_0,\xi)}$, we have $C_{L^*\pm}(t,x,\xi) = \overline{C_{L\pm}(t,x,\xi)}$. Here $^{-}$ denotes the complex conjugate. Therefore, the assumption (1) also holds for L^* and we have a pair of fundamental solutions $V_0(t,s)$, $V_1(t,s)$, $t \leqq s \leqq 0$, of the Cauchy problem for L^* in $t \leqq s \leqq 0$, $t=s$ being the initial surface. Let f_0, f_1 be any distributions in $\mathcal{E}'(\boldsymbol{R}^n)$ and set $u(t) = U_0^-(t,s)f_0 + U_1^-(t,s)f_1$. Similarly, for arbitrary $g_0, g_1 \in C^\infty(\boldsymbol{R}^n)$, we set $v(t) = V_0(t,s)g_0 + V_1(t,s)g_1$. Consider the identity:

$$\int_t^s \langle Lu(\tau), v(\tau)\rangle d\tau - \int_t^s \langle u(\tau), L^*v(\tau)\rangle d\tau = 0.$$

This means, by the integrations by parts, that

$$\Psi(t,s) = \begin{pmatrix} I & 0 \\ b_0(s,\cdot)I & I \end{pmatrix} \begin{pmatrix} D_tV_1(t,s)^* & V_1(t,s)^* \\ D_tV_0(t,s)^* & V_0(t,s)^* \end{pmatrix} \begin{pmatrix} I & I \\ b_0(t,\cdot)I & I \end{pmatrix},$$

where $*$ stands for the adjoint.

16. Changing the variables, we set, for $\tau \leqq t \leqq 0$,

$$\Psi(\tau,t) = \begin{pmatrix} \Psi_0(\tau,t) & \Psi_1(\tau,t) \\ \Psi_0'(\tau,t) & \Psi_1'(\tau,t) \end{pmatrix}.$$

Then, by § 14, $\Psi_0(\tau,t)$, $\Psi_1(\tau,t)$ coincide with the fundamental solutions to the problem $(*)$ when $\tau \leqq t < 0$. Furthermore, if

$$w(t) = \Psi_0(\tau,t)f_0 + \Psi_1(\tau,t)f_1,$$

then by lemma $w(0-)$ and $D_tw(0-)$ are well-defined. Set

$$w'(t) = U_0^+(t,0)w(0-) + U_1^+(t,0)D_tw(0-)$$

for $t>0$. Then $w'(0+)=w(0-)$, $D_t w'(0+)=D_t w(0-)$, and, by the equation, $D_t^2 w'(0+)=D_t^2 w(0-)$ and so forth. Therefore, setting for $j=0,1$,

$$U_j(t,\tau)=U_j^+(t,\tau) \qquad \text{if } t\geqq\tau\geqq 0,$$

and

$$U_j(t,\tau)=\begin{cases}\Psi_j(\tau,t) & \text{if } 0>t\geqq\tau,\\ U_0^+(t,0)\Psi_j(\tau,0)+U_1^+(t,0)D_t\Psi_j(\tau,0) & \text{if } t\geqq 0\geqq\tau,\end{cases}$$

we obtain a pair of fundamental solutions to the problem (*) in $t\geqq\tau$.

17. As we already remarked in § 15, the formal adjoint L^* of L also satisfies the assumption (1). This implies uniqueness of the pair $U_0(t,\tau)$, $U_1(t,\tau)$.

18. Further details and generalizations as well as consequences of Theorem will be discussed elsewhere. Note that the present treatment is akin to that of Oleinik [6]. Compare her Theorem 2 [6] and our Lemma in § 13.

References

[1] Alinhac, S.: Paramétrix pour un système hyperbolique à multiplicité variable. Comm. Partial Diff. Eq., **2**, 251–296 (1977).

[2] Chazarain, J.: Opérateurs hyperboliques à caractéristiques de multiplicité constante. Ann. Inst. Fourier, **24**, 173–202 (1974).

[3] Ivrii, V. Ya., and V. M. Petkov: Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic operators. Usp. Mat. Nauk, **29**, 3–70 (1974).

[4] Hörmander, L.: The Cauchy problem for differential equations with double characteristics (to appear).

[5] Ohya, Y.: Le problème de Cauchy à caractéristiques multiples. C. R. Acad. Sc. Paris, **238**, Sér. A, 1433–1436 (1976).

[6] Oleinik, O. A.: On the Cauchy problem for weakly hyperbolic equations. Comm. Pure Appl. Math., **23**, 569–586 (1970).

[7] Yoshikawa, A.: Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation. I. Hokkaido Math. J., **6**, 313–344 (1977).

[8] ——: Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation. II, III. Hokkaido Math. J. (in press).