Notes on geometric aspects of effectively hyperbolic critical points when the characteristic roots are real on one side of time

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Abstract: Geometric aspects of effectively hyperbolic critical points on time $t = 0$ are discussed assuming that the characteristic roots are real on one side of time t, namely time is positive. In particular, we aim to elucidate the differences in the geometric aspects of effectively hyperbolic critical points on time $t = 0$ when the characteristic roots are real on both the positive and negative sides of time.

Key words: Hamilton map; Time function; Effectively hyperbolic critical point; Cauchy problem.

1. Introduction. Consider

$$P = -D_t^2 + A_2(t,x,D) + \sum_{j=0}^{1} A_j(t,x,D)D_t^{1-j} \tag{1.1}$$

where $A_j(t,x,D), D = (D_1,\dots,D_d), D_k = -i\partial/\partial x_k$ are differential operators of order j with coefficients smooth in a neighborhood of $(t,x) = (0,0) \in \mathbf{R}^{1+d}$. Denote the principal symbol by

$$p(t,x,\tau,\xi) = -\tau^2 + a(t,x,\xi)$$

where $a(t,x,\xi)$ is homogeneous of degree 2 in ξ, smooth in $(-T,T) \times U \times \mathbf{R}^d$ and satisfies

$$a(t,x,\xi) \geq 0, \quad (t,x,\xi) \in [0,T_1) \times U \times \mathbf{R}^d \tag{1.2}$$

with some $T_1 > 0$ and a neighborhood U of $0 \in \mathbf{R}^d$. Note that if $(0,0,\tau,\xi)$, $(\tau,\xi) \neq 0$ is a critical point of $p = 0$ then $\tau = 0$ and $a(0,0,\xi) = 0$. A critical point $\rho = (0,0,0,\bar{\xi})$, $\bar{\xi} \neq 0$ is called effectively hyperbolic if the Hamilton map (e.g. [2, 1])

$$F_p = \frac{1}{2}\begin{pmatrix} \partial^2 p/\partial x\partial\xi & \partial^2 p/\partial\xi\partial\xi \\ -\partial^2 p/\partial x\partial x & -\partial^2 p/\partial\xi\partial x \end{pmatrix}$$

has nonzero real eigenvalues at ρ. Assuming a slightly stronger assumption than (1.2) such that

$$a(t,x,\xi) \geq 0, \quad (t,x,\xi) \in (-\delta_1,T_1) \times U \times \mathbf{R}^d$$

with some $\delta_1 > 0$, a critical point ρ is effectively hyperbolic if and only if one can find a smooth function $\varphi(x,\xi)$ in some conic neighborhood V of $(0,\bar{\xi})$, homogeneous of degree 0 in ξ, and constants $0 < \kappa < 1$, $c > 0$, $\delta > 0$ such that

$$a(t,x,\xi) \geq c(t-\varphi(x,\xi))^2|\xi|^2, \quad \{\varphi, a\}^2 \leq 4\kappa a$$

for $|t| < \delta$ and $(x,\xi) \in V$ ([4, Lemma 1.2.2]) where $\{\varphi, a\}$ denotes the Poisson bracket of φ and a. In [6, 4] this is the key to proving that the Cauchy problem for P with Cauchy data on $t = \tau$ for small $|\tau|$ is C^∞ well-posed for arbitrary lower order terms.

This note aims to provide a proof for the following proposition [7, Proposition 13.1], which was used in [7] without being proved.

Proposition 1.1. *Assume* (1.2). *If a critical point $\rho = (0,0,0,\bar{\xi})$ of $p = 0$ is effectively hyperbolic then there exist a smooth function $\varphi(x,\xi)$ in a conic neighborhood V of $(0,\bar{\xi})$, homogeneous of degree 0 in ξ, and constants $0 < \kappa < 1$, $c > 0$, $\delta > 0$ such that*

$$a(t,x,\xi) \geq c\min\{t^2, (t-\varphi(x,\xi))^2\}|\xi|^2, \tag{1.3}$$

$$\{\varphi, a\}^2 \leq 4\kappa a \tag{1.4}$$

for $(t,x,\xi) \in [0,\delta) \times V$. Conversely, if there exists such a φ the critical point ρ is effectively hyperbolic.

Note that the condition (1.4) with $\kappa < 1$ implies that $f = t - \varphi(x,\xi)$ is a time function at $\rho = (0,0,0,\bar{\xi})$ for p in the following sense (1.5); denote the quadratic part of $a(t,x,\xi+\bar{\xi})$ by $\bar{a}(t,x,\xi)$ such that $p(t,x,\tau,\xi+\bar{\xi}) = -\tau^2 + \bar{a}(t,x,\xi) + O(|(t,x,\xi)|^3)$ then denoting $p_\rho = -\tau^2 + \bar{a}(t,x,\xi)$ we have

$$p_\rho(-H_f(\rho)) < 0 \tag{1.5}$$

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where $H_f = (0, -\partial\varphi/\partial\xi, -1, \partial\varphi/\partial x)$ is the Hamilton vector field of f. To check (1.5), noting that (1.4) is invariant under symplectic change of coordinates, one can assume either $\varphi = x_1$ or $d\varphi(0,\bar\xi) = 0$ and $\bar\xi = e_d$. In the latter case (1.5) is obvious. When $\varphi = x_1$ the assertion is also clear if $\bar a(0,x,\xi)$ contains no ξ_1. Otherwise one can write $\bar a(0,x,\xi) = c(\xi_1 - h(x,\xi'))^2 + g(x,\xi')$ where $\xi' = (\xi_2,\dots,x_d)$ with $c \neq 0$. From (1.4) we have $0 < c \le \kappa < 1$ which proves (1.5).

Under the assumption (1.2) the characteristic roots may not be real on $t < 0$ side. Consider $p = -\tau^2 + a(t,x)\xi^2$ with $a(t,x) = (t-x)^2 + x^3$ where $x \in \mathbf{R}$, $|x| < 1$. It is easy to check that (1.2) is verified and ± 1 are the eigenvalues of the Hamilton map at $(0,0,0,1)$ while $a(t,x)$ can be negative on $t < 0$ side in any small neighborhood of $(0,0) \in \mathbf{R}^2$.

Applying Proposition 1.1 one can prove the following

Theorem 1.1. *In order that the Cauchy problem for P such that finding a solution on $t \ge 0$ side with Cauchy data given at $t = 0$, is C^∞ well-posed for arbitrary lower order terms it is necessary and sufficient that* (1.2) *holds and every critical point* $(0,0,0,\xi)$, $\xi \neq 0$ *of* $p = 0$ *is effectively hyperbolic.*

To prove the sufficient part we apply the pseudodifferential weight with symbol $w = t\phi(t,x,\xi)$, similar as in [7, Section 10], where $\phi = \omega + t - \varphi$ and $\omega^2 = (t-\varphi)^2 + \langle\xi\rangle^{-1}$ with φ obtained in Proposition 1.1. Compared to the triple characteristic case [7, Section 6], there is no vanishing factor before $\langle\xi\rangle^{-1}$, so the condition (1.4) is required to estimate the commutator of the weight and $A_2(t,x,D)$ (roughly sketched in [4, Section 2]). The necessary part of the theorem has been proved in [2]. If, near every critical point, p admits a decomposition $p = q_1q_2$ with real smooth symbols q_i vanishing at the reference point the sufficient part has also been proved in [3], but this assumption implies, in particular, the characteristic roots are real and smooth even on the $t < 0$ side near the reference point.

2. Proof of Proposition 1.1.

2.1. Preliminary lemmas. In what follows we denote $x^{(p)} = (x_p,\dots,x_d)$, $\xi^{(p)} = (\xi_p,\dots,\xi_d)$, $1 \le p \le d$ and $t = x_0$.

Lemma 2.1. *Assume* (1.2) *and that* $(0,0,0,\bar\xi)$ *is a critical point such that* $\partial_t^2 a(0,0,\bar\xi) \neq 0$. *Then there is a homogeneous symplectic coordinates system* (x,ξ) *around* $(0,\bar\xi)$ *such that the coordinates of* $(0,\bar\xi)$ *is* $(0,e_d)$ *and* $a(t,x,\xi)$ *takes the form either* $(1)_p$ *with* $(1c)_p$ $(0 \le p \le d-1)$;

$$(1)_p \quad \begin{aligned}&\sum_{i=1}^p (x_{i-1}-x_i)^2 q_i(t,x,\xi) + \sum_{i=1}^p \xi_i^2 r_i(t,x,\xi)\\ &\quad + \{(x_p - \phi_p(x^{(p+1)},\xi^{(p+1)}))^2\\ &\quad + \psi_p(x^{(p+1)},\xi^{(p+1)})\} q_{p+1}(t,x,\xi),\end{aligned}$$

$$(1c)_p \qquad \{\phi_p, \{\phi_p, \psi_p\}\}(0,0,e_d) = 0$$

or $(2)_p$ *with* $(2c)_p$ $(1 \le p \le d-1)$;

$$(2)_p \quad \begin{aligned}&\sum_{i=1}^p (x_{i-1}-x_i)^2 q_i(t,x,\xi) + \sum_{i=1}^p \xi_i^2 r_i(t,x,\xi)\\ &\quad + g_p(x^{(p)},\xi^{(p+1)}) r_p(t,x,\xi),\end{aligned}$$

$$(2c)_p \qquad \{\xi_p, \{\xi_p, g_p\}\}(0,0,e_d) = 0.$$

In both cases $q_i(t,x,\xi)$, $r_i(t,x,\xi)$ *are homogeneous of degree* 2, 0 *respectively in* ξ, *positive at* $(0,0,e_d)$ *and* ϕ_p, ψ_p, g_p *are homogeneous of degree* 0, 0, 2 *respectively in* ξ *vanishing at* $(0,0,e_d)$.

Proof. It is enough to repeat exactly the same arguments as the proof of [5, Lemma 2.1] with minor changes. After a linear change of coordinates x one can assume that $(0,\bar\xi) = (0,e_d)$. Since $\partial_t^2 a(0,0,e_d) \neq 0$ thanks to the Malgrange preparation theorem one can write

$$\begin{aligned}a(t,x,\xi) = \{(t - \phi_0(x^{(1)},\xi^{(1)}))^2 + \psi_0(x^{(1)},\xi^{(1)})\}\\ \times q_1(t,x^{(1)},\xi^{(1)})\end{aligned}$$

where ϕ_0, ψ_0 are homogeneous of degree 0 vanishing at $(0,e_d)$ and q_1 is homogeneous of degree 2 with $q_1(0,0,e_d) \neq 0$. Since $a(t,0,e_d) = t^2 q_1(t,0,e_d)$ it follows that $q_1(0,0,e_d) > 0$. Hence if $\{\phi_0,\{\phi_0,\psi_0\}\}(0,e_d) = 0$ this is just $(1)_0$ with $(1c)_0$. We go on to the induction on p. Assume that $(1)_{p-1}$ holds while $(1c)_{p-1}$ fails. Set $X_p(x^{(p)},\xi^{(p)}) = \phi_{p-1}(x^{(p)},\xi^{(p)})$. Following the proof of [5, Lemma 2.1] one can find a homogeneous symplectic coordinates $\{X_j(x^{(p)},\xi^{(p)}),\ \Xi_j(x^{(p)},\xi^{(p)})\}_{j=p}^d$ such that

$$(2.1) \qquad \begin{aligned}&X_j(0,e_d) = 0,\ p \le j \le d,\\ &\Xi_j(0,e_d) = 0,\ p \le j \le d-1,\ \Xi_d(0,e_d) \neq 0.\end{aligned}$$

Denoting $\{X_j,\ \Xi_j\}_{j=p}^d$ by $\{x_j,\xi_j\}_{j=p}^d$ again and noting that $\partial_{\xi_p}^2 \psi_{p-1}(0,e_d) \neq 0$ one can write

$$\begin{aligned}\psi_{p-1}(x^{(p)},\xi^{(p)}) = \{(\xi_p - h_p(x^{(p)},\xi^{(p+1)}))^2\\ + g_p(x^{(p)},\xi^{(p+1)})\} b_p(x^{(p)},\xi^{(p)})\end{aligned}$$

by use of the Malgrange preparation theorem where b_p is of homogeneous of degree -2 with $b_p(0,e_d) \neq 0$ and h_p, g_p are homogeneous of degree 1, 2 respectively, vanishing at $(0,e_d)$. Take $x=0$ and $\xi_j=0$ unless $j=p,d$ and $\xi_d=1$ then we have

$$a(0,0,\xi)/\xi_p^2 = b_p(0,\xi)q_p(0,0,\xi) \geq 0$$

which implies $b_p(0,e_d)q_p(0,0,e_d) \geq 0$ hence we have $b_p(0,e_d) > 0$ for $q_p(0,0,e_d) > 0$. Set

$$\Xi_p = \xi_p - h_p(x^{(p)}, \xi^{(p+1)}), \quad X_p = x_p.$$

Again following the proof of [5, Lemma 2.1] we can extend Ξ_p, X_p to a homogeneous symplectic coordinates $\{X_j(x^{(p)}, \xi^{(p)}), \Xi_j(x^{(p)}, \xi^{(p)})\}_{j=p}^d$ verifying (2.1), where $\xi_j(X^{(p)}, \Xi^{(p)})$, $p+1 \leq j \leq d$ and $x_j(X^{(p)}, \Xi^{(p)})$, $p \leq j \leq d$ are independent of Ξ_p. Thus we obtain $(2)_p$ with $r_p = b_p q_p$ which is positive at $(0,0,e_d)$. Now assume that $(2c)_p$ fails. Then one can write

$$g_p(x^{(p)}, \xi^{(p+1)}) = \{(x_p - \phi_p(x^{(p+1)}, \xi^{(p+1)}))^2 + \psi_p(x^{(p+1)}, \xi^{(p+1)})\}c_p(x^{(p)}, \xi^{(p+1)})$$

where c_p is homogeneous of degree 2 with $c_p(0,e_d) \neq 0$ and ϕ_p, ψ_p are homogeneous of degree 0, vanishing at $(0,e_d)$. Choose $x^{(p+1)} = 0$, $\xi = e_d$ and $x_p = \cdots = x_1 = x_0 (= t) > 0$ then

$$a(t,x,\xi)/x_p^2 = c_p(x_p, 0, e_d) r_p(t,x,\xi) \geq 0$$

hence c_p is positive at $(0,e_d)$ because $r_p(0,0,e_d) > 0$ and so is $q_{p+1} = c_p r_p$ at $(0,0,e_d)$. Thus we conclude that $(1)_p$ holds. Therefore the induction on p proves the assertion. □

Lemma 2.2. *Assume that* (1.2) *holds. If the critical point* $(0,0,0,e_d)$ *is effectively hyperbolic then* $\partial_t^2 a(0,0,e_d) \neq 0$ *and* $a(t,x,\xi)$ *has the form either* $(1)_p$ *with* $(1c)_p$ *or* $(2)_p$ *with* $(2c)_p$ *and*

$$\sum_{i=1}^p r_i^{-1}(0,0,e_d) > 1. \tag{2.2}$$

The converse is also true.

Proof. If $\partial_t^2 a(0,0,e_d) = 0$ the Taylor expansion of $a(t,x,\xi+e_d)$ at $(t,x,\xi) = (0,0,0)$ gives

$$a(\epsilon^2 t, \epsilon^3 x, \epsilon^3 \xi + e_d) = \epsilon^5 t \sum_{|\alpha+\beta|=1} \partial_t \partial_x^\alpha \partial_\xi^\beta a(0,0,e_d) x^\alpha \xi^\beta + O(\epsilon^6)$$

as $\epsilon \to 0$ which proves that

$$\partial_t \partial_x^\alpha \partial_\xi^\beta a(0,0,e_d) = 0, \quad |\alpha+\beta| = 1 \tag{2.3}$$

since the left-hand side is nonnegative for small $t \geq 0$, $|\epsilon|$ and for (x,ξ) near $(0,0)$. Then it is easy to see that $F_p(0,0,0,e_d)$ has only pure imaginary eigenvalues, contradicting to that $(0,0,0,e_d)$ is effectively hyperbolic, hence $\partial_t^2 a(0,0,e_d) \neq 0$ so that Lemma 2.1 can be applied. To complete the proof it suffices to follow the considerations in [5, Section 2]. Assume $(1)_p$ with $(1c)_p$. If $d\phi_p(0,e_d) \neq 0$ one may assume $\phi_p = x_{p+1}$ so that $\psi_p(x_{p+1}, x^{(p+2)}, \xi^{(p+1)}) \geq 0$ for any $(x_{p+1}, x^{(p+2)}, \xi^{(p+1)})$ with $x_{p+1} \geq 0$ near $(0,e_d)$ hence the Hessian of ψ_p at $(0,e_d)$ is nonnegative definite then $\partial^2\psi_p/\partial x_j \partial\xi_{p+1} = \partial^2\psi_p/\partial\xi_j\partial\xi_{p+1} = 0$ at $(0,e_d)$ for any j because $\partial_{\xi_{p+1}}^2 \psi_p(0,e_d) = 0$ by $(1c)_p$. Therefore the characteristic polynomial $\det(\lambda I - F_p)$ is a product of two characteristic polynomials (if $d\phi_p(0,e_d) = 0$ this is clear) and we conclude that F_p has nonzero real eigenvalues ([5, Section 2]). Assume $(2)_p$ with $(2c)_p$. From the same argument as above the Hessian of g_p at $(0,e_d)$ is nonnegative definite and $\partial_{x_p}^2 g_p(0,e_d) = 0$ which implies that $\partial^2 g_p/\partial x_j \partial x_p = \partial^2 g_p/\partial\xi_j\partial x_p = 0$ at $(0,e_d)$ for any j. Then it follows from [5, Section 2] that F_p has nonzero real eigenvalues if and only if the condition (2.2) holds. □

2.2. Proof of Proposition 1.1. Without restrictions we may assume that $(0,\bar{\xi}) = (0,e_d)$. Taking into account the homogeneity in ξ it suffices to show (1.3), (1.4) in a small neighborhood of $(0,e_d)$. First consider the case $(1)_p$. Denote $x_a = (x_1,\ldots,x_p)$, $x_b = (x_{p+1},\ldots,x_d)$ and $\xi_a = (\xi_1,\ldots,\xi_p)$, $\xi_b = (\xi_{p+1},\ldots,\xi_d)$ and, by an obvious abuse of notation, we often write $q(t,x,\xi) = q(t,x_a,\xi_a,x_b,\xi_b)$. Let $\chi(s) \in C^\infty(\mathbf{R})$ be such that $\chi(s) = s$ for $|s| \leq 1$, $|\chi(s)| = 2$ for $|s| \geq 2$ and $\chi'(s) \geq 0$. By replacing x_k, ξ_k by $\delta\chi(x_k/\delta)$, $\delta\chi(\xi_k/\delta)$, $k = 1,\ldots,p$ with a small $\delta > 0$ in $q_j(t,x,\xi)$, $r_j(t,x,\xi)$ we may assume that such obtained ones, which we denote by the same q_j, r_j, are defined for all $(x_a,\xi_a) \in \mathbf{R}^{2p}$. Considering a/q_{p+1} we may assume that $q_{p+1} = 1$. Writing $(x_b,\xi_b) = z + (0,e_d)$, $w = (y_a,\eta_a)$ and $\theta = (t,z,\varepsilon)$, we consider

$$Q(w,\theta) = Q(w,t,z,\varepsilon) = \sum_{j=1}^p (y_{j-1}-y_j)^2 q_j(t, \varepsilon y_a + t1_a, \varepsilon\eta_a, z + (0,e_d)) + (y_p+1)^2 + \sum_{j=1}^p \eta_j^2\, r_j(t, \varepsilon y_a + t1_a, \varepsilon\eta_a, z+(0,e_d))$$

where $y_0=0$ and $1_a=\overbrace{(1,\ldots,1)}^{p}$. If we choose $\varepsilon=t-\phi(z)$ with $\phi(z)=\phi_p(z+(0,e_d))$ then

$$a(t,\varepsilon y_a+t1_a,\varepsilon\eta_a,z+(0,e_d))=\varepsilon^2 Q(w,\theta)+\psi(z)$$

with $\psi(z)=\psi_p(z+(0,e_d))$. Denote $\bar q_j=q_j(0,0,e_d)$, $\bar r_j=r_j(0,0,e_d)$ and note that

$$Q(w,0)=\sum_{j=1}^{p}(y_{j-1}-y_j)^2\bar q_j+(y_p+1)^2+\sum_{j=1}^{p}\eta_j^2\bar r_j$$

$(y_0=0)$ takes a positive minimum value in $\mathbf{R}^{2p}$. For any $\epsilon>0$ one can choose $\delta>0$ such that $\big|Q(w,\theta)-Q(w,0)\big|\le\epsilon\, Q(w,0)$ for all $w\in\mathbf{R}^{2p}$ if $|t|+|z|<\delta$ because $|\delta\chi(s/\delta)|\le 2\delta$ for any $s\in\mathbf{R}$. Therefore for small $|\theta|$, $Q(w,\theta)$ takes a positive minimum in $w\in\mathbf{R}^{2p}$ which is achieved in $|w|<B$ with some $B>0$ independent of small θ. Denote

$$\min_{w\in\mathbf{R}^{2p}}Q(w,\theta)=m(\theta).$$

Note that the Hessian $\nabla_w^2 Q(w,\theta)$ of $Q(w,\theta)$ with respect to w can be written $\nabla_w^2 Q(w,\theta)=H+R(w,\theta)$ with a nonsingular constant matrix H. Here for any $\epsilon_1>0$ there exist $\delta>0$, $\delta_1>0$ such that $\|R(w,\theta)\|\le\epsilon_1$ if $|\theta|<\delta_1$ and $|w|<B$ since $|(d/ds)^j\delta\chi(\varepsilon s/\delta)|\le C\varepsilon^j\delta^{j-1}$, $j=1,2$. Therefore thanks to the implicit function Theorem there exists a smooth $\bar w(\theta)$ near $\theta=(0,0,0)$ such that

$$m(\theta)=Q(\bar w(\theta),\theta).$$

Taking $\varepsilon=t-\phi(z)$ we have

$$\begin{aligned}&a(t,\varepsilon y_a+t1_a,\varepsilon\eta_a,z+(0,e_d))\\ &=(t-\phi(z))^2Q(w,t,z,t-\phi(z))+\psi(z)\\ &\ge m(t,z,t-\phi(z))(t-\phi(z))^2+\psi(z)\\ &=m_1(t,z)(t-\phi(z))^2+\psi(z)\end{aligned}\tag{2.4}$$

where $m_1(t,z)=m(t,z,t-\phi(z))$. Assume $\phi(z)<0$ and hence $\varepsilon=t-\phi(z)>0$ for $t\ge0$. Then choosing $w=(y_a,\eta_a)$ so that $(\varepsilon y_a+t1_a,\varepsilon\eta_a)=(x_a,\xi_a)$ one concludes that

$$\begin{aligned}a(t,x,\xi)&=a(t,\varepsilon y_a+t1_a,\varepsilon\eta_a,z+(0,e_d))\\ &\ge m_1(t,z)(t-\phi(z))^2+\psi(z).\end{aligned}$$

Moreover choosing $w=\bar w(t,z,t-\phi(z))$ in (2.4) it follows that

$$m_1(t,z)(t-\phi(z))^2+\psi(z)\ge0,\quad t\ge0.$$

In particular, taking $t=0$ we have

$$m_1(0,z)\phi^2(z)+\psi(z)\ge0.\tag{2.5}$$

Noting that $m_1(t,z)\ge c_1>0$ we also have

$$\begin{aligned}a(t,x,\xi)&\ge m_1(t,z)\big(t-\phi(z)\big)^2+\psi(z)\\ &=m_1(t,z)\big(t^2+2t|\phi(z)|+\phi^2(z)\big)+\psi(z)\\ &\ge c_1t^2+2c_1t|\phi(z)|+m_1(0,z)\phi^2(z)+\psi(z)\\ &\quad+\big(m_1(t,z)-m_1(0,z)\big)\phi^2(z).\end{aligned}$$

Since $|m_1(t,z)-m_1(0,z)|\le Ct$ it follows from (2.5) that

$$\begin{aligned}&a(t,x,\xi)\ge c_1t^2\\ &+t|\phi(z)|\big(2c_1-C|\phi(z)|\big)\ge c_1t^2\end{aligned}\tag{2.6}$$

for $t\ge0$ and $\phi(z)<0$ near $z=0$ because $\phi(0)=0$. Next if $\phi(z)\ge0$ then choosing $x_j=\phi(z)\ge0$, $j=0,1,\ldots,p$ and $\xi_j=0$, $j=1,\ldots,p$ in $(1)_p$ it follows that $\psi(z)\ge0$ hence we have

$$a(t,x,\xi)\ge\big(t-\phi(z)\big)^2,\quad\phi(z)\ge0.\tag{2.7}$$

Thus from (2.6) and (2.7) we conclude that

$$a(t,x,\xi)\ge c\,\min\{t^2,(t-\phi(z))^2\}|\xi|^2,\quad t\ge0.$$

Next, recalling $\phi(z)=\phi_p(x_b,\xi_b)$ it is clear that $\{\phi,\{\phi,a\}\}(0,0,e_d)=\{\phi_p,\{\phi_p,\psi_p\}\}(0,0,e_d)=0$ by $(1c)_p$. Then for any $\epsilon_1>0$ one can find a neighborhood U of $(0,e_d)$ such that

$$\big|H_\phi^2\,a\big|=\big|\{\phi,\{\phi,a\}\}\big|\le\epsilon_1,\quad(x,\xi)\in U$$

for small t. Since $a\ge0$ for $t\ge0$ thanks to Glaeser's inequality we see that

$$\big|H_\phi a\big|^2=\big|\{\phi,a\}\big|^2\le2\,\epsilon_1\,a,\quad t\ge0$$

which finishes the proof for the case $(1)_p$.

Turn to case $(2)_p$. Denote $x_a=(x_1,\ldots,x_{p-1})$, $x_b=(x_{p+1},\ldots,x_d)$ and $\xi_a=(\xi_1,\ldots,\xi_p)$, $\xi_b=(\xi_{p+1},\ldots,\xi_d)$. As above we extend q_j, r_j, replacing x_k, $1\le k\le p-1$ and ξ_k, $1\le k\le p$ by $\delta\chi(x_k/\delta)$ and $\delta\chi(\xi_k/\delta)$ so that such extended ones are defined for all $(x_a,\xi_a)\in\mathbf{R}^{p-1}\times\mathbf{R}^p$. Considering a/r_p we may assume $r_p=1$ as before. Denoting $\theta=(t,z,x_p,\varepsilon)$ consider

$$\begin{aligned}&Q(w,\theta)=Q(w,t,z,x_p,\varepsilon)\\ &=(y_1+1)^2q_1(t,x_p1_a-\varepsilon y_a,x_p,\varepsilon\eta_a,z+(0,e_d))\\ &+\sum_{j=1}^{p-1}(y_j-y_{j+1})^2q_{j+1}(t,x_p1_a-\varepsilon y_a,x_p,\varepsilon\eta_a,z+(0,e_d))\\ &+\sum_{j=1}^{p}\eta_j^2\,r_j(t,x_p1_a-\varepsilon y_a,x_p,\varepsilon\eta_a,z+(0,e_d)),\ y_p=0\end{aligned}$$

where $w=(y_a,\eta_a)\in\mathbf{R}^{2p-1}$ and $(x_b,\xi_b)=z+(0,e_d)$ as before. Note that if we choose $\varepsilon=t-x_p$ then

$$a(t, x_p 1_a - \varepsilon y_a, x_p, \varepsilon\eta, z + (0, e_d)) = \varepsilon^2 Q(w, \theta) + g(x_p, z)$$

where $g(x_p, z) = g_p(x_p, x_b, \xi_b)$. Noting that

$$Q(w, 0) = (y_1+1)^2\bar{q}_1 + \sum_{j=1}^{p-1}(y_j - y_{j+1})^2\bar{q}_{j+1} + \sum_{j=1}^{p}\eta_j\bar{r}_j$$

($y_p = 0$) one can repeat a similar argument as above to conclude that there is $\bar{w}(\theta)$ smooth near $\theta = 0$ such that

$$\min_{w\in\mathbf{R}^{2p-1}} Q(w, \theta) = m(\theta) = Q(\bar{w}(\theta), \theta).$$

Choosing $\varepsilon = t - x_p$ we have

$$\begin{aligned} &a(t, x_p 1_a - \varepsilon y_a, x_p, \varepsilon\eta_a, z + (0, e_d)) \\ &= (t - x_p)^2 Q(w, t, z, x_p, t - x_p) + g(x_p, z) \\ &\geq m(t, z, x_p, t - x_p)(t - x_p)^2 + g(x_p, z) \\ &= m_1(t, x_p, z)(t - x_p)^2 + g(x_p, z) \end{aligned} \tag{2.8}$$

where we have set $m_1(t, x_p, z) = m(t, z, x_p, t - x_p)$. When $x_p < 0$, repeating the same arguments as above one can find $c_1 > 0$

$$a(t, x, \xi) \geq c_1 t^2 + t|x_p|(2c_1 - C|x_p|) \geq c_1 t^2 \tag{2.9}$$

for $x_p < 0$ and $t \geq 0$ in a neighborhood of $(0, e_d)$. Assume $x_p \geq 0$. Thanks to Lemma 2.2 one has $\sum_{i=1}^{p} 1/\bar{r}_i > 1$ then one can find $\epsilon_i > 0$ such that $\sum_{i=1}^{p}\epsilon_i^2\,\bar{r}_i = \rho < 1$ with $\sum_{i=1}^{p}\epsilon_i = 1$. Define

$$\phi(x) = \sum_{i=1}^{p}\epsilon_i x_i.$$

Since $x_{i-1} - x_i = 0$, $i = 1, \ldots, p$ implies $t - \phi(x) = x_0 - x_0 = 0$ hence $t - \phi(x)$ is a linear combination of $x_{i-1} - x_i$ then there is $C > 0$ such that

$$(t - \phi(x))^2 \leq C\sum_{i=1}^{p}(x_{i-1} - x_i)^2 q_i.$$

Taking $x_p = t \geq 0$ in (2.8) one has $g(x_p, z) \geq 0$ for $x_p \geq 0$ which proves $a(t, x, \xi) \geq \sum_{i=1}^{p}(x_{i-1} - x_i)^2 q_i$ for $x_p \geq 0$. From this there is $c' > 0$ such that

$$a(t, x, \xi) \geq c'(t - \phi(x))^2, \quad x_p \geq 0. \tag{2.10}$$

Thus from (2.9) and (2.10) it follows that

$$a(t, x, \xi) \geq c\,\min\{t^2, (t - \phi(x))^2\}|\xi|^2, \quad t \geq 0.$$

It is clear that for any $\epsilon > 0$ there is a neighborhood U of $(0, e_d)$ such that

$$|H_\phi^2 a| \leq 2\sum_{i=1}^{p}\epsilon_i^2\,\bar{r}_i + \epsilon = 2\rho + \epsilon, \quad (x, \xi) \in U$$

for small t because $H_\phi^2 a(0, 0, e_d) = 2\sum_{i=1}^{p}\epsilon_i^2\,\bar{r}_i$. Since $a \geq 0$ for $t \geq 0$ it follows from Glaeser's inequality that

$$|H_\phi a|^2 = |\{\phi, a\}|^2 \leq 2(2\rho + \epsilon)a, \quad (x, \xi) \in U$$

for small $t \geq 0$ where one can assume $2\rho + \epsilon < 2$ for $\rho < 1$. This completes the proof for the case $(2)_p$.

2.3. Proof of Proposition 1.1 (continued). Assume that there is a smooth $\varphi(x, \xi)$, homogeneous of degree 0 in ξ, satisfying (1.3), (1.4). Without restrictions one can assume that $\bar{\xi} = e_d$. Note that $\partial_t^2 a(0, 0, e_d) \neq 0$. If not from (2.3) the quadratic part $\bar{a}$ of $a(t, x, \xi + e_d)$ contains no t which clearly contradicts (1.3). Thus one can apply Lemma 2.1 to conclude that $a(x_0, x, \xi)$ has the form either $(1)_p$ with $(1c)_p$ or $(2)_p$ with $(2c)_p$. If a has the form $(1)_p$ with $(1c)_p$ then $(0, 0, 0, e_d)$ is effectively hyperbolic by virtue of Lemma 2.2. Assume that a has the form $(2)_p$ with $(2c)_p$. Denote

$$\varphi(x, \xi + e_d) = \sum_{j=1}^{d}(c_j x_j + c_j'\xi_j) + O(|(x, \xi)|^2)$$

with $c_j, c_j' \in \mathbf{R}$. Taking $t = x_0 = x_1 = \cdots = x_p > 0$, $x^{(p+1)} = (x_{p+1}, \ldots, x_d) = 0$, $\xi = e_d$ in $(2)_p$ we obtain from (1.3) that

$$\begin{aligned} &g_p(x_p, 0, e_d)r_p(x, e_d) \\ &\geq c\min\{x_p^2, (x_p - \varphi(x_p 1_a, 0, e_d))^2\}. \end{aligned}$$

Since $g_p(x_p, 0, e_d) = O(x_p^3)$ by $(2c)_p$ one concludes

$$\sum_{j=1}^{p} c_j = 1. \tag{2.11}$$

Next, when $(t, x, \xi) = (0, \xi_a, e_d)$ and $|\xi_a| \to 0$, it is easy to see that

$$a = \sum_{j=1}^{p}\xi_j^2\bar{r}_j + O(|\xi_a|^3),$$

$$\{\varphi, a\} = -2\sum_{j=1}^{p}c_j\xi_j\bar{r}_j + O(|\xi_a|^2).$$

Letting $|\xi_a| \to 0$ in $\{\varphi, a\}^2/|\xi_a|^2 \leq 4\kappa a/|\xi_a|^2$ we obtain

$$\Big(\sum_{j=1}^{p}c_j\omega_j\bar{r}_j\Big)^2 \leq \kappa\sum_{j=1}^{p}\omega_j^2\bar{r}_j, \quad |(\omega_1, \ldots, \omega_p)| = 1.$$

Choosing $\omega_j = 1/(R\bar{r}_j)$ with $R^2 = \sum_{j=1}^{p}1/\bar{r}_j^2$ we conclude that $1 \leq \kappa\sum_{j=1}^{p}1/\bar{r}_j$ hence the assertion follows from Lemma 2.2.

References

[1] L. Hörmander, The Cauchy problem for differential equations with double characteristics, J. Anal. Math., **32** (1977), 118–196.

[2] V. Ivrii and V. M. Petkov, Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations, Uspehi Mat. Nauk, **29** (1974), no. 5, 3–70.

[3] V. Ivrii, Sufficient conditions for regular and completely regular hyperbolicity, Trudy Moskov. Mat. Obšč., **33** (1975), 3–65.

[4] K. Kajitani and T. Nishitani, *The hyperbolic Cauchy problem*, Lecture Notes in Math., 1505, Springer-Verlag, Berlin, 1991, pp. 71-167.

[5] T. Nishitani, A note on reduced forms of effectively hyperbolic operators and energy integrals, Osaka J. Math., **21** (1984), no. 4, 843–850.

[6] T. Nishitani, Local energy integrals for effectively hyperbolic operators. I, II, J. Math. Kyoto Univ., **24** (1984), no. 4, 623–658, 659–666.

[7] T. Nishitani, The Cauchy problem for operators with triple effectively hyperbolic characteristics: Ivrii's conjecture, J. Anal. Math., **149** (2023), no. 1, 167–237.