88. On the Adiabatic Theorem for the Hamiltonian System of Differential Equations in the Classical Mechanics. I

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Introduction. In this paper, we shall use the following abbreviations:

$$p=(p_1,\cdots,p_n),\ q=(q_1,\cdots,q_n),\ (p,q)=(p_1,\cdots,p_n,q_1,\cdots,q_n)$$
$$dp/dt=(dp_1/dt,\cdots,dp_n/dt),\ dq/dt=(dq_1/dt,\cdots,dq_n/dt),$$
$$\partial H/\partial p=(\partial H/\partial p_1,\cdots,\partial H/\partial p_n),\ \partial H/\partial q=(\partial H/\partial q_1,\cdots,\partial H/\partial q_n)$$
$$dpdq=dp_1\cdots dp_n\cdot dq_1\cdots dq_n.$$

Let $H(p,q,s)$ be a Hamiltonian containing a parameter s $(1\geqq s\geqq 0)$. Roughly speaking, a quantity $I(p,q,s)$ which is a global integral of the system

$$(1)\qquad dp/dt=-\partial H/\partial q(p,q,s),\ dq/dt=\partial H/\partial p(p,q,s)$$

for every fixed s $(1\geqq s\geqq 0)$, is called an adiabatic invariant of (1), if $I(p,q,t/\lambda)$ is conserved along all (nearly all)[1] trajectories of the following system

$$(2)\qquad dp/dt=-\partial H/\partial q(p,q,t/\lambda),\ dq/dt=\partial H/\partial p(p,q,t/\lambda)$$

in the whole interval of time $0\leqq t\leqq\lambda$, asymptotically for $\lambda\to+\infty$. The fundamental case of the adiabatic theorem in the classical mechanics is the case where (1) has no one-valued Lebesgue measurable global integral other than the functions of the energy integral for almost all s $(1\geqq s\geqq 0)$.[2] In this case, the phase volume

$$\mathfrak{J}(p,q,s)=\int_{I_{E,s}} dpdq$$

where $I_{E,s}$ means the domain in (p,q)-space enclosed by the energy surface $S_{E,s}=\{(p,q)\mid H(p,q,s)=E\}$ of (1) passing through (p,q), is one (and essentially the only one)[3] adiabatic invariant. In the following, we shall call this proposition the adiabatic theorem.

The adiabatic theorem plays an important rôle not only in the statistical mechanics, but also in various other branches of physics. But as far as we know, satisfactory proofs of the theorem exist

1) Here the word "nearly all" is used in a vague sense.

2) The cases where this assumption does not hold, can be reduced to this fundamental case if the reductions as given in T. Levi-civita [7] are possible and the reduced system satisfies an assumption similar to this assumption. Cf. T. Levi-civita [7], also H. Geppert [1].

3) The "only one" part of the adiabatic theorem shall be treated in Part IV of this paper.

only for very special cases where essentially only one pair of canonical variables (p_1, q_1) occurs.[4)5)] In this paper, we shall give a precise formulation of the adiabatic theorem in the general case and prove it by the aid of the operator method in the classical mechanics of J. v. Neumann and B. O. Koopman.[6)]

For convenience sake, we change the scale of time in (2) putting $t'=t/\lambda$ and discuss the asymptotic behaviour for $\lambda\to+\infty$ of the trajectories of the transformed system

(3) $dp/dt'=-\lambda\partial H/\partial q(p, q, t')$ $dq/dt'=\lambda\partial H/\partial q(p, q, t')$

in the interval $1\geqq t'\geqq 0$ instead of that of (2) in the interval $\lambda\geqq t\geqq 0$.

In Part I of this paper, we shall determine the region D in R^{2n+1} such that the adiabatic theorem shall be stated for the trajectories of (3) lying in it and shall study the properties of the phase volume function $\widetilde{\mathfrak{J}}(p, q, s)$ in D. In Part II, we shall study the flows (Strömungen in the sense of J. v. Neumann) defined by (1) and the infinitesimal generator iA of the unitary group $\{U_t\mid -\infty<t<+\infty\}$ associated with one of the flows.[6)] In Part III, we shall state and prove a form of the adiabatic theorem (in Section 8), under Assumptions 1 and 2 stated in Section 1 and Assumption 3 stated in Section 5. In Part IV, we shall state and prove a more satisfactory form of the adiabatic theorem and some discussions of the results obtained shall follow it. Also in Part IV, we shall prove some lemmas and theorems stated but unproved in Parts I and II.

Notations. If B is a subset of a Euclidian space or a subset of a Hilbert space, we denote by $\overline{B}$, the closure of B in the space and by B^0 the set of all inner points of B in the space. In this paper a function is always complex-valued, if not specially mentioned. We denote by $C^m(B)$, the set of all m times continuously differentiable functions on B if B is a subset of a Euclidian space such that $\overline{B^0}\supset B$ and by $C_0^m(B)$ the set of all m times continuously differentiable functions on B vanishing outside compact sets contained in B, if further B is open in the space. Also we write $C(B), C_0(B)$ for $C^0(B), C_0^0(B)$.

1. We shall call a subset of a subset B of a Euclidian space *relatively open* in B if it can be considered as an intersection of an open set in the Euclidian space with B.

We denote by K the point set $\{(p, q, s)\mid -\infty<p_i<+\infty,\ -\infty<q_i<+\infty\ i=1,\cdots, n, a\leqq s\leqq b\}$ in R^{2n+1} where a and b are two fixed

4) For these special cases, cf. H. Kneser [4], A. Lenard [6], Y. Watanabe [10].

5) The arguments in T. Levi-civita [7] only make the holding of the adiabatic theorem for the general case probable. He and other authors proved various generalizations of Theorem 1 of this paper but they did not prove the adiabatic theorem itself for the general case. Cf. T. Levi-civita [7], H. Geppert [1], G. Mattioli [8].

6) Cf. J. v. Neumann [9], B. O. Koopman [5] and E. Hopf [2].

real numbers such that $a<b$.

ASSUMPTION 1. *The point set in R^{2n+1} where the real-valued function $H(p,q,s)$ is defined, contains a subset G of K relatively open in K satisfying the following two conditions*:

i) *H, $\partial H/\partial s$, $\partial H/\partial p_i$, $\partial H/\partial q_i$, $\partial^2 H/\partial p_i \partial q_j$, $\partial^2 H/\partial s \partial p_i$, $\partial^2 H/\partial s \partial q_i$ $(i, j=1, \cdots, n)$ exist and are continuous on G. Also*

$$\sum_{i=1}^{n}\{(\partial H/\partial p_i)^2+(\partial H/\partial q_i)^2\}\neq 0 \text{ on } G.$$

ii) *We denote by G_s the open set $\{(p,q)\,|\,(p,q,s)\in G\}$ in (p,q)-space and by $S_{E,s}$ the set $\{(p,q)\,|\,H(p,q,s)=E,\ (p,q)\in G_s\}$ in G_s. Then for each s $(a\leqq s\leqq b)$, there is an open interval $\Gamma_s=\{E\,|\,E_s<E<E'_s\}$ $(-\infty \leqq E_s<E'_s=+\infty)$ on E-line such that $S_{E,s}$ is non-void and is a closed $(2n-1)$-dimensional C^1-submanifold of R^{2n} enclosing a domain $I_{E,s}$ in R^{2n} for each $E\in\Gamma_s$ and $G_s=\bigcup_{E\in\Gamma_s} S_{E,s}$.*

Under Assumption 1, we can prove after some topological considerations that two alternative cases occur: A) $I_{E_2,s}\supset S_{E_1,s}$ for all E_2, E_1, s such that $a\leqq s\leqq b$, $E_s<E_1<E_2<E'_s$ or B) $I_{E_1,s}\supset S_{E_2,s}$ for all E_1, E_2, s such that $a\leqq s\leqq b$, $E_s<E_1<E_2<E'_s$. We shall assume in the following that the former case A) occurs. The later case B) can be treated just in the same way. Then we can also prove the following two consequences iii), iv) of Assumption 1.

iii) *$I_{E_2,s}\supset \bar{I}_{E_1,s}=I_{E_1,s}\cup S_{E_1,s}$ for all E_1, E_2, s such that $a\leqq s\leqq b$, $E_s<E_1<E_2<E'_s$. We put $I_{E_1,E_2,s}=I_{E_2,s}-\bar{I}_{E_1,s}$ for such E_1, E_2, s. Then $I_{E_1,E_2,s}=\bigcup_{E_1<E<E_2} S_{E,s}$ and*

$$\bar{I}_{E_1,E_2,s}=I_{E_1,E_2,s}\cup S_{E_1,s}\cup S_{E_2,s}$$

and $\bar{I}_{E_1,E_2,s}$ is compact. The point set $\Lambda=\{(E,s)\,|\,a\leqq s\leqq b,\ E_s<E<E'_s\}$ in (E,s)-plane is relatively open in the point set $\{(E,s)\,|\,a\leqq s\leqq b, -\infty<E<+\infty\}$ in (E,s)-plane.

iv) *If we put*

$$\mathfrak{J}(E,s)=\int_{I_{E,s}} dpdq,$$

then $\mathfrak{J}(E,s)\in C^1(\Lambda)$ and $\partial\mathfrak{J}/\partial E>0$ for $(E,s)\in\Lambda$.

We omit the proof of these results and assume the results since such considerations are not the main purposes of this paper.

By iv), if we put $J_s=\lim_{E\to E_s}\mathfrak{J}(E,s)$ and $J'_s=\lim_{E\to E'_s}\mathfrak{J}(E,s)$, then the point set $\Lambda'=\{(J,s)\,|\,a\leqq s\leqq b, J_s<J<J'_s\}$ is relatively open in the point set $\{(J,s)\,|\,a\leqq s\leqq b, -\infty<J<+\infty\}$ in the (J,s)-plane and if $(J,s)\in\Lambda'$, the equation $J=\mathfrak{J}(E,s)$ can be solved uniquely for E. If we denote the solution by $E=\mathfrak{E}(J,s)$, $\mathfrak{E}(J,s)\in C^1(\Lambda')$ and $\partial\mathfrak{E}/\partial J>0$ for $(J,s)\in\Lambda'$. Also if we put $\tilde{\mathfrak{J}}(p,q,s)=\mathfrak{J}(H(p,q,s),s)$, then we have from i) of Assumption 1

$$\tilde{\mathfrak{J}}(p,q,s)\in C^1(G),\ \sum_{i=1}^{n}\{(\partial\tilde{\mathfrak{J}}/\partial p_i)^2+(\partial\tilde{\mathfrak{J}}/\partial q_i)^2\}\neq 0$$

on G.

ASSUMPTION 2. *There are two numbers J_1^*, J_2^* independent of s such that $J_s'>J_2^*>J_1^*>J_s$ for all s in the interval $a\leqq s\leqq b$.*

In the following, we shall fix such a pair of J_1^* and J_2^* once for all. We put $S(J,s)=S_{\mathfrak{G}(J,s),s}=\{(p,q)\,|\,(p,q)\in G_s,\ \tilde{\mathfrak{J}}(p,q,s)=J\}$, $I(J,s)=I_{\mathfrak{G}(J,s),s}$, $I(J_1,J_2,s)=I_{\mathfrak{G}(J_1,s),\,\mathfrak{G}(J_2,s),s}=\{(p,q)\,|\,(p,q)\in G_s,\ J_2>\tilde{\mathfrak{J}}(p,q,s)>J_1\}$ for J, J_1, J_2, s, such that $a\leqq s\leqq b, J_2^*\geqq J\geqq J_1^*$ and $J_2^*\geqq J_2>J_1\geqq J_1^*$. We also put $I(s)=I(J_1^*,J_2^*,s)=\{(p,q)\,|\,(p,q)\in G_s\ J_1^*<\tilde{\mathfrak{J}}(p,q,s)<J_2^*\}$, $D=\{(p,q,s)\,|\,0\leqq s\leqq b,(p,q)\in I(s)\}=\{(p,q,s)\,|\,(p,q,s)\in G, J_2^*>\tilde{\mathfrak{J}}(p,q,s)>J_1^*\}$, $D(J)=\{(p,q,s)\,|\,a\leqq s\leqq b,(p,q)\in I(J,s)\}$, $W(J)=\{(p,q,s)\,|\,a\leqq s\leqq b,(p,q)\in S(J,s)\}=\{(p,q,s)\,|\,(p,q,s)\in G,\ \tilde{\mathfrak{J}}(p,q,s)=J\}$ for s, J such that $a\leqq s\leqq b$, $J_1^*\leqq J\leqq J_2^*$. Then D is relatively open in K since $\tilde{\mathfrak{J}}(p,q,s)\in C^1(G)$. Also we can prove the following consequence v) of Assumptions 1 and 2.

v) *$D(J)$ is relatively open in K and $\overline{D(J)}, W(J), \overline{D}$ are compact and $\overline{D(J)}=D(J)\cup W(J), \overline{D}=D\cup W(J_2^*)\cup W(J_2^*)$ for $J_1^*\leqq J\leqq J_2^*$.*

We omit the proof of v) and assume v) by the same reason as before.

2. In this paper, a function is always complex-valued if not specially mentioned.

A measure space (X,m) is a pair of a set X and a measure m defined on a Borel field in X. When X is a Lebesgue measurable subset of a Euclidian space R^r and m is the usual Lebesgue measure in R^r defined for all Lebesgue measurable subset of X, we shall often omit the explicit indication of the measure m and for example a function on X is simply called *measurable* or *integrable on X* in the following if it is measurable or integrable on the measure space (X,m).

LEMMA 1. *There is a unique measure $m_{J,s}$ defined for all Borel sets*[7] *on each $S(J,s)(a\leqq s\leqq b, J_2^*>J>J_1^*)$ such that*

$$\frac{d}{dJ}\int_{I(J_1^*,J,s)}f(p,q)dpdq=\int_{S(J,s)}f(p,q)dm_{J,s} \tag{4}$$

for every function $f(p,q)\in C_0[I(s)]$. We denote the completion of $m_{J,s}$ also by $m_{J,s}$. Then for any integrable (or non-negative measurable) function $f(p,q)$ on $I(s)$, $f(p,q)$ is integrable (or non-negative measurable) on the measure space $(S(J,s),m_{J,s})$ for almost all J ($J_2^*>J>J_1^*$) and

7) A Borel set on $S(J,s)$ is a subset of $S(J,s)$ belonging to the Borel field on $S(J,s)$ generated by all relatively open subsets of $S(J,s)$.

$$\int_{S(J,s)} f(p,q)dm_{J,s}$$

is integrable (or non-negative measurable) with respect to J in the interval $J_2^* > J > J_1^*$ and

$$(5) \qquad \int_{I(s)} f(p,q)dpdq = \int_{J_1^*}^{J_2^*} \Big(\int_{S(J,s)} f(p,q)dm_{J,s} \Big) dJ.$$

Also

$$(6) \qquad \int_{S(J,s)} dm_{J,s} = 1.$$

We shall give a proof of this lemma in Part IV.

Now we give a rigorous proof of the following theorem which is well known in the classical statistical mechanics.

THEOREM 1.

$$\int_{S(J,s)} \frac{\partial \tilde{\mathfrak{J}}}{\partial s}(p,q,s)dm_{J,s} = 0$$

for $J_2^* > J > J_1^*$ and $b \geqq s \geqq a$.

PROOF. We take any function $\varphi(J) \in C_0[(J_1^*, J_2^*)]$ and put

$$G(J) = -\int_J^{J_2^*} \varphi(J')dJ' \quad \text{for} \quad J_2^* > J > J_1^*.$$

Then $G(J) \in C^1[(J_1^*, J_2^*)]$, $dG/dJ = \varphi(J)$ and there exist two numbers J_1 and J_2 such that $J_2^* > J_2 > J_1 > J_1^*$ and

$$G(J) = 0 \quad \text{for} \quad J_2^* > J \geqq J_2$$
$$G(J) = G(J_1) \quad \text{for} \quad J_1 \geqq J > J_1^*.$$

If we define a function $F(p,q,s)$ on K by

$$F(p,q,s) = G(\tilde{\mathfrak{J}}(p,q,s)) \quad \text{for} \quad (p,q,s) \in D$$
$$F(p,q,s) = 0 \quad \text{for} \quad K - D(J_2^*)$$
$$F(p,q,s) = G(J_1) \quad \text{for} \quad \overline{D(J_1^*)},$$

then by the consequence v) of Assumptions 1 and 2 in Section 1, $F(p,q,s)$ belongs to $C^1(K)$ and vanishes for all (p,q,s) with sufficiently large $|p|^2 + |q|^2$.

Now by Lemma 1 and the properties of $G(J)$ and the definition of $F(p,q,s)$, we have for $a \leqq s \leqq b$

$$\int_{R^{2n}} F(p,q,s)dpdq = \int_{I(s)} G(\tilde{\mathfrak{J}}(p,q,s))dpdq + \int_{I(J_1^*,s)} G(J_1)dpdq$$
$$= \int_{J_1^*}^{J_2^*} \Big(\int_{S(J,s)} G(\tilde{\mathfrak{J}}(p,q,s))dm_{J,s} \Big) dJ + J_1^* \cdot G(J_1) = \int_{J_1^*}^{J_2^*} G(J)dJ + J_1^* \cdot G(J_1).$$

Hence we have for $a \leqq s \leqq b$

$$(7) \qquad \frac{d}{ds} \int_{R^{2n}} F(p,q,s)dpdq = 0.$$

On the other hand, also by Lemma 1, we have for $a \leqq s \leqq b$

$$
(8)\qquad
\begin{aligned}
\frac{d}{ds}\int_{R^{2n}} F(p,q,s)dpdq &= \int_{R^{2n}} \frac{\partial}{\partial s}F(p,q,s)dpdq \\
&= \int_{I(s)} \frac{dG}{dJ}(\tilde{\mathfrak{J}}(p,q,s))\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dpdq \\
&= \int_{J_1^*}^{J_2^*}\Big(\int_{S(J,s)} \frac{dG}{dJ}(\tilde{\mathfrak{J}}(p,q,s))\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ \\
&= \int_{J_1^*}^{J_2^*}\frac{dG}{dJ}\Big(\int_{S(J,s)}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ.
\end{aligned}
$$

From (7) and (8), we have

$$
\int_{J_1^*}^{J_2^*}\varphi(J)\Big(\int_{S(J,s)}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ=0
$$

for all s ($a\leqq s\leqq b$) and all $\varphi(J)\in C_0[(J_1^*, J_2^*)]$. From this, by a well-known standard argument we can get the desired result. Q. E. D.

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