79. Three Primes in Arithmetical Progression

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1. One of the long-standing conjectures on the distribution of prime numbers states that there are infinitely many m-plets of primes $p_1, p_2, \cdots, p_m$ in arithmetical progression for every $m>2$, which is, at least, empirically true. Unfortunately, however, we cannot at present prove (or disprove) the existence of such an m-plet of primes for an unspecified value of the number m. Some sequences of prime numbers are known to be in arithmetical progression. For example, the ten numbers

$$119+210\,n \qquad (n=0, 1, 2, \cdots, 9)$$

are all primes (cf. [1]).

Our aim in the present note is to show that there exist infinitely many triplets of primes p_1, p_2, p_3 in arithmetical progression, i.e. such that $p_1<p_3$ and

$$p_1+p_3=2p_2.$$

In fact, we can prove somewhat more. Let a be a positive integer, b an arbitrary integer, and let $N(x, a, b)$ denote the number of solutions of

$$p_1+p_3=ap_2+b$$

in prime numbers p_1, p_2, p_3 with $2\leqq p_j\leqq x$ $(j=1, 2, 3)$. Then there holds the following

Theorem. *We have*

$$N(x, a, b)=C(a, b)\ T(x, a, b)+O(x^2(\log x)^{-A}) \quad (x\to\infty)$$

for every $A>3$, where the O-constant depends possibly on a, b and A and where

$$C(a, b)=\prod_{p\mid a,\, p\mid b}\frac{p}{p-1}\prod_{\substack{p\mid a,\, p\nmid b\\ \text{or}\\ p\nmid a,\, p\mid b}}\frac{p(p-2)}{(p-1)^2}\prod_{p\nmid ab}\left(1+\frac{1}{(p-1)^3}\right);$$

$$T(x, a, b)=\sum(\log n_1 \log n_2 \log n_3)^{-1},$$

the summation being extended over all integer solutions n_1, n_2, n_3 of the equation

$$n_1+n_3=an_2+b$$

with $2\leqq n_j\leqq x$ $(j=1, 2, 3)$.

It is easy to see from our result that $C(a, b)>0$ unless a and b have a different parity and, in particular, we have

$$C(2, 0)\geqq 2(\zeta(2))^{-1}=\frac{12}{\pi^2}.$$

Also, it should be noted that

$$T(x, a, b) > \frac{1}{2a}\frac{x^2}{(\log x)^3} \qquad (x > x_0(a, b)).$$

2. We shall give only a sketchy proof of our theorem. There is a standard method due to I. M. Vinogradov [3] of trigonometrical sums by means of which one can deal with problems on prime numbers of the kind here considered. Thus we define the function

$$S(t)=\sum_{p\leq x} e(pt),$$

where p runs through the primes and $e(t)=\exp 2\pi it$. We have

$$N(x, a, b)=\int_I (S(t))^2 S(-at)e(-bt)dt$$

for any interval I of unit length.

Put for a positive integer q

$$G(q)=\frac{(\mu(q))^2 c_q(a)c_q(b)}{(\phi(q))^3},$$

where $c_q(n)$ denotes Ramanujan's sum, i. e.

$$c_q(n)=\sum_{\substack{0\leq h<q \\ (h, q)=1}} e\left(\frac{hn}{q}\right)=\sum_{d\mid(n, q)} d\mu\left(\frac{q}{d}\right).$$

Let B be an arbitrary but fixed real number greater than $2A+7$ and set $Q=(\log x)^B$. Then, just as in [2], we deduce from the above identity for $N(x, a, b)$ the following asymptotic equation:

$$N(x, a, b)=\sum_{1\leq q\leq Q} G(q)T(x, a, b)+O(x^2(\log x)^{-A}).$$

Now we have for any $\varepsilon>0$

$$G(q)=C(a, b)+O((\log x)^{-B(1-\varepsilon)}),$$

on noticing that $G(q)$ is a multiplicative function of q, i. e. that $(q_1, q_2)=1$ implies $G(q_1q_2)=G(q_1)G(q_2)$.

It follows that

$$N(x, a, b)=C(a, b)\, T(x, a, b)+O(T(x, a, b)(\log x)^{-B(1-\varepsilon)})$$
$$+O(x^2(\log x)^{-A}),$$

and this completes the proof of our theorem since

$$T(x, a, b)=O(x^2(\log x)^{-3}).$$

References

[1] L. E. Dickson: History of the Theory of Numbers, **1**, New York (1934). Especially, Chap. XVIII, 425–426.

[2] K. Prachar: Primzahlverteilung, Springer-Verl., Berlin-Göttingen-Heidelberg (1957). Especially, Chap. VI.

[3] I. M. Vinogradov: The method of trigonometrical sums in the theory of numbers (in Russian), Trav. Inst. Math. Stekloff, **23** (1947).