

60. *On the fundamental differential equations of flat projective geometry.*

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§ 1. In an address given at the International Mathematical Congress at Bologna,¹⁾ Prof. O. Veblen defined the geometry as the theory of an invariant. According to the Erlanger Programm a geometry is the invariant theory of a group. According to the new conception of O. Veblen a geometry is the theory of invariant.

O. Veblen showed, in a lecture to the London Mathematical Society,²⁾ that the classical projective geometry may be regarded as the theory of an invariant subject to certain restrictions. The theory of this invariant free from these restrictions is the so-called generalized projective geometry.

If we introduce a curvilinear coordinate system $(\xi^i)^{3)}$ in an n -dimensional projective space, the homogeneous projective coordinates Z^λ of the space may be expressed in the form

$$(1.1) \quad Z^\lambda = e^{\lambda 0} p_\mu^\lambda f^\mu(\xi^i),$$

where $e^{\lambda 0}$ is an arbitrary factor, p_μ^λ constants subject only to the condition that the determinant $|p_\mu^\lambda|$ formed with p_μ^λ is different from zero and finally $f^\mu(\xi^i)$ $n+1$ analytic functions of ξ^i such that the determinant

$$(1.2) \quad \begin{vmatrix} f^0 & f^1 & \dots & f^n \\ \frac{\partial f^0}{\partial \xi^1} & \frac{\partial f^1}{\partial \xi^1} & \dots & \frac{\partial f^n}{\partial \xi^1} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f^0}{\partial \xi^n} & \frac{\partial f^1}{\partial \xi^n} & \dots & \frac{\partial f^n}{\partial \xi^n} \end{vmatrix} \neq 0.$$

Differentiating (1.1) and eliminating the constants, we find that any $n+1$ homogeneous projective coordinates Z^λ defined as functions of curvilinear coordinates ξ^i must satisfy the differential equations

(1) O. Veblen: Differential invariants and geometry. *Atti Congresso Internazionale Bologna*, **1**, (1928), 181-189.

(2) O. Veblen: Generalized projective geometry. *Journal of the London Math. Soc.*, **4** (1929), 140-160.

(3) Greek indices take the values on the range $0, 1, 2, \dots, n$ and Roman indices on the range $1, 2, \dots, n$.

$$(1.3) \quad \frac{\partial^2 Z}{\partial \xi^\mu \partial \xi^\nu} = \Gamma_{\mu\nu}^\lambda \frac{\partial Z}{\partial \xi^\lambda},$$

where $\Gamma_{\mu\nu}^\lambda$ are functions of $\xi^1, \xi^2, \dots, \xi^n$ only and satisfy the following conditions

$$(1.4) \quad \Gamma_{0\nu}^\lambda = \Gamma_{\nu 0}^\lambda = \delta_{\nu}^\lambda,$$

$$(1.5) \quad \Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda,$$

$$(1.6) \quad B_{\mu\nu\omega}^\lambda \equiv \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial \xi^\omega} - \frac{\partial \Gamma_{\mu\omega}^\lambda}{\partial \xi^\nu} + \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\omega}^\lambda - \Gamma_{\mu\omega}^\alpha \Gamma_{\alpha\nu}^\lambda = 0.$$

The condition (1.4) follows from the special form of (1.1). The last two are integrability conditions of the differential equations (1.3).

The equations (1.3) are the so-called fundamental differential equations of flat projective geometry.

As we may multiply the homogeneous projective coordinates by any factor and may change the curvilinear coordinate system to any other one, the equations (1.3) must be unaltered in the form by the transformations

$$(1.7) \quad \begin{cases} \bar{\xi}^0 = \xi^0 + \log \rho(\xi^1, \xi^2, \dots, \xi^n), \\ \bar{\xi}^i = \xi^i(\xi^1, \xi^2, \dots, \xi^n) \end{cases}$$

By these transformations, the functions $\Gamma_{\mu\nu}^\lambda$ are transformed into $\bar{\Gamma}_{\mu\nu}^\lambda$, where

$$(1.8) \quad \bar{\Gamma}_{\mu\nu}^\lambda = \frac{\partial \bar{\xi}^\lambda}{\partial \xi^\alpha} \left(\frac{\partial \xi^\beta}{\partial \bar{\xi}^\mu} \frac{\partial \xi^\tau}{\partial \bar{\xi}^\nu} \Gamma_{\beta\tau}^\alpha + \frac{\partial^2 \xi^\alpha}{\partial \bar{\xi}^\mu \partial \bar{\xi}^\nu} \right).$$

The conditions (1.4), (1.5) and (1.6) are invariant by these transformations.

As the transformation law (1.8) is transitive, we may define an invariant whose components in the coordinate system (ξ^λ) are $\Gamma_{\mu\nu}^\lambda$ and whose components $\bar{\Gamma}_{\mu\nu}^\lambda$ in the coordinate system $(\bar{\xi}^\lambda)$ are given by (1.8). This invariant is called the projective connection.

The theory of an arbitrary projective connection, whose components are functions of n curvilinear coordinates and satisfy the conditions (1.4) and (1.5), is a generalized projective geometry. If the components of a projective connection satisfy the conditions (1.4), (1.5) and (1.6), the theory of this projective connection is flat projective geometry.

It is to be noticed that, if the components satisfy (1.4), (1.5) and (1.6), these components are functions of $\xi^1, \xi^2, \dots, \xi^n$ only. In fact, we have, from (1.6),

$$B_{\mu\nu 0}^\lambda = \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial \xi^0} - \frac{\partial \Gamma_{\mu 0}^\lambda}{\partial \xi^\nu} + \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha 0}^\lambda - \Gamma_{\mu 0}^\alpha \Gamma_{\alpha\nu}^\lambda = 0,$$

from which

$$\frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial \xi^{\sigma}} = 0.$$

§ 2. Let us consider (1.3) as the fundamental differential equations of an $n+1$ dimensional affine space A_{n+1} and $\Gamma_{\mu\nu}^{\lambda}$ as the components of affine connection of the space.

If there exists, in A_{n+1} , a coordinate system with respect to which the components $\Gamma_{\mu\nu}^{\lambda}$ of the affine connection satisfy the conditions (1.4), (1.5) and (1.6), then the affine space A_{n+1} , referred to this coordinate system, may be taken to represent the projective space P_n .

Following J. H. C. Whitehead,⁴⁾ we shall denote any coordinate system for A_{n+1} by K_{n+1} and a K_{n+1} which is a representation for P_n by R_n .

Let

$$(2.1) \quad Z^{\lambda} = e^{\xi^0} f^{\lambda}(\xi^i)$$

be a set of $n+1$ independent solutions of the differential equations. The fact that the solutions are of the form (2.1) is ensured by the conditions (1.4). If we consider the equations (1.3) as the fundamental differential equations of the $(n+1)$ -dimensional flat affine space A_{n+1} and $\Gamma_{\mu\nu}^{\lambda}$ as the components of the affine connection, the solutions Z^{λ} given by (2.1) are rectilinear affine coordinates of the space, and with respect to this coordinate system, all the components of the affine connection vanish.

Thus, the rectilinear coordinate system (Z^{λ}) is an K_{n+1} for the affine space, but it is not a representation for the projective space P_n , that is, it is not an R_n . This is due to the fact that we are using the non-homogeneous coordinate system but not the homogeneous one.

To obtain an R_n , we put

$$(2.2) \quad \begin{cases} z^0 = \log Z^0, \\ z^i = Z^i / Z^0, \end{cases}$$

or

$$(2.3) \quad \begin{cases} z^0 = \xi^0 + \log f^0(\xi^1, \xi^2, \dots, \xi^n), \\ z^i = f^i(\xi^j). \end{cases}$$

The transformation of coordinates $\xi^{\lambda} \rightarrow z^{\lambda}$ is of the form (1.7). If we denote by $P_{\mu\nu}^{\lambda}$ the components of the projective connection, we have, as it may be easily verified,

$$(2.4) \quad P_{\nu}^{\lambda} = P_{\nu 0}^{\lambda} = \delta_{\nu}^{\lambda} \quad \text{and} \quad P_{jk}^{\lambda} = 0.$$

4) J.H.C. Whitehead: On a class of projectively flat affine connections. Proc. London Math. Soc., **32** (1931), 93-114; The representation of projective spaces. Annals of Math., **32** (1931), 327-360.

Thus the coordinate system (z^λ) is an R_n .

If we regard Z^λ as a rectilinear coordinates in A_{n+1} , the relation between (Z^λ) and (z^λ) is given by

$$(2.5) \quad \begin{cases} Z^0 = e^{z^0}, \\ Z^i = e^{z^0} z^i. \end{cases}$$

Thus the points of P_n , which are represented, in A_{n+1} , by the straight lines through the origin, are represented, in (z^λ) , by the curves of parametes z^0 .

§ 3. In the flat case, the finite equations of the paths, say, of the straight lines are given by

$$(3.1) \quad Z^\lambda = A^\lambda + B^\lambda t,$$

where A^λ and B^λ are homogeneous coordinates of two points on the straight line. Differentiating this equation twice with respect to t and taking account of the fundamental differential equations, we find

$$(3.2) \quad \frac{d^2 \xi^\lambda}{dt^2} + \Gamma_{\mu\nu}^\lambda \frac{d\xi^\mu}{dt} \frac{d\xi^\nu}{dt} = 0$$

as the differential equations of the paths.

Putting $\lambda=0$ and $\lambda=i$ respectively in (3.2), we have

$$(3.3) \quad \begin{cases} (i) & \frac{d^2 \xi^0}{dt^2} + \left(\frac{d\xi^0}{dt}\right)^2 + \Gamma_{jk}^0 \frac{d\xi^j}{dt} \frac{d\xi^k}{dt} = 0, \\ (ii) & \frac{d^2 \xi^i}{dt^2} + \Gamma_{jk}^i \frac{d\xi^j}{dt} \frac{d\xi^k}{dt} + 2 \frac{d\xi^0}{dt} \frac{d\xi^i}{dt} = 0 \end{cases}$$

in virtue of the relations $\Gamma_{00}^\lambda = \Gamma_{j0}^\lambda = \delta_{j\lambda}$. If we put

$$2 \frac{d\xi^0}{dt} = \frac{d^2 t}{ds^2} \left(\frac{dt}{ds}\right)^2$$

or

$$(3.4) \quad \xi^0 = \frac{1}{2} \log \frac{dt}{ds},$$

we have, from (3.3) (i),

$$(3.5) \quad \{t, s\} = -2 \Gamma_{jk}^0 \frac{d\xi^j}{ds} \frac{d\xi^k}{ds},$$

and, from (3.3) (ii),

$$(3.6) \quad \frac{d^2 \xi^i}{ds^2} + \Gamma_{jk}^i \frac{d\xi^j}{ds} \frac{d\xi^k}{ds} = 0,$$

where $\{t, s\}$ is the Schwarzian derivative of t with respect to s , s being an affine parameter on paths. Thus the parameter t is a projective one.

If we take the coordinate system (z^λ) defined in the last Paragraph, the equations (3.5) and (3.6) gives

$$(3.7) \quad t = \frac{\alpha s + \beta}{\gamma s + \delta}$$

and

$$(3.8) \quad z^t = a^t s + b^t$$

respectively, where $\alpha, \beta, \gamma, \delta, a^t$ and b^t are constants.

The parameter t introduced in the equation of path (3.1) is a projective one. If we use an arbitrary parameter r on path, the finite equation of the paths is

$$(3.9) \quad Z^\lambda = A^\lambda u(r) + B^\lambda v(r).$$

Differentiating twice this equation with respect to r , and eliminating constants from these equations, we find the differential equations of the form

$$(3.10) \quad \frac{d^2 Z^\lambda}{dr^2} = \alpha \frac{dZ^\lambda}{dr} + \beta Z^\lambda.$$

The homogeneous coordinates Z^λ being functions of curvilinear coordinates ξ^λ , we have, from (3.10),

$$(3.11) \quad \frac{d^2 \xi^\lambda}{dr^2} + \Gamma_{\mu\nu}^\lambda \frac{d\xi^\mu}{dr} \frac{d\xi^\nu}{dr} = \alpha \frac{d\xi^\lambda}{dr} + \beta \delta_0^\lambda$$

in virtue of the fundamental differential equations and of the relation $Z^\lambda = \frac{\partial Z^\lambda}{\partial \xi^\nu}$.

The equations (3.11) are the most general equations of the paths.

If we regard Z^λ or ξ^λ as the coordinates of points in an $(n+1)$ -dimensional affine space A_{n+1} , the equations (3.1) are those of straight lines passing by two points A^λ and B^λ . The points A^λ and B^λ of projective space P_n being represented by a straight lines in A_{n+1} joining these points to origin, we may say that the paths in P_n are represented by a plane in A_{n+1} containing three points A^λ, B^λ and the origin.

Thus, the paths in P_n being represented by planes passing through the origin in A_{n+1} , to represent the paths, we may take arbitrary curves on these planes passing through the origin. The equations (3.11) are equations of such curves. Indeed, the equations show that, if we regard ξ^λ as coordinate of points in an $(n+1)$ -dimensional affine space A_{n+1} , the osculating plane of this curve contains always the vector δ_0^λ . The vector whose components are δ_0^λ being tangent to the ξ^0 -curves or rays, it passes through always the origin.

§ 4. In Paragraph 1, we have assumed the homogeneous projective coordinates Z^λ of an n -dimensional projective space to be of the form (1.1), ξ^0 and ξ^t being an arbitrary factor and curvilinear coordinates of the space respectively.

Let us now introduce, in an n -dimensional projective space P_n , a system of

curvilinear homogeneous coordinates (x^λ) and assume that the homogeneous projective coordinates Z^λ of the space are expressed in the form

$$(4.1) \quad Z^\lambda = p_\mu^\lambda F^\mu(x^\alpha),$$

where p_μ^λ are constants subject only to the conditions that the determinant $|p_\mu^\lambda|$ formed with them is different from zero and $F^\lambda(x^\alpha)$ are $n+1$ homogeneous functions of degree $+1$ such that

$$(4.2) \quad \begin{vmatrix} \frac{\partial F^0}{\partial x^0}, & \frac{\partial F^1}{\partial x^0}, & \dots, & \frac{\partial F^n}{\partial x^0} \\ \frac{\partial F^0}{\partial x^1}, & \frac{\partial F^1}{\partial x^1}, & \dots, & \frac{\partial F^n}{\partial x^1} \\ \dots & \dots & \dots & \dots \\ \frac{\partial F^0}{\partial x^n}, & \frac{\partial F^1}{\partial x^n}, & \dots, & \frac{\partial F^n}{\partial x^n} \end{vmatrix} \neq 0.$$

Differentiating (4.1) and eliminating the constants p_μ^λ , we find the so-called fundamental differential equations

$$(4.3) \quad \frac{\partial^2 Z}{\partial x^\mu \partial x^\nu} = \Pi_{\mu\nu}^\lambda \frac{\partial Z}{\partial x^\lambda},$$

which any $n+1$ homogeneous projective coordinates Z^λ defined as the functions of $n+1$ curvilinear homogeneous coordinates x^λ must satisfy.

The functions Z^λ being homogeneous of degree $+1$ with respect to x^α , the coefficients $\Pi_{\mu\nu}^\lambda$ are homogeneous functions of the degree -1 with respect to x^α .

The functions Z^λ being homogeneous of degree $+1$, the functions $\frac{\partial Z}{\partial x^\lambda}$ are homogeneous of degree 0 , so that if we multiply (4.3) by x^λ and sum up for the index ν , we find

$$0 = \Pi_{\mu\nu}^\lambda x^\nu \frac{\partial Z}{\partial x^\lambda},$$

from which we obtain

$$(4.5) \quad \Pi_{\mu\nu}^\lambda x^\nu = 0,$$

the $n+1$ points being linearly independent in virtue of (4.2).

The integrability conditions of the differential equations (4.3) are

$$(4.6) \quad \Pi_{\mu\nu}^\lambda = \Pi_{\nu\mu}^\lambda,$$

$$(4.7) \quad \Omega_{\mu\nu\omega}^\lambda \equiv \frac{\partial \Pi_{\mu\nu}^\lambda}{\partial x^\omega} - \frac{\partial \Pi_{\mu\omega}^\lambda}{\partial x^\nu} + \Pi_{\mu\nu}^\alpha \Pi_{\alpha\omega}^\lambda - \Pi_{\mu\omega}^\alpha \Pi_{\alpha\nu}^\lambda = 0.$$

As we may change the curvilinear homogeneous coordinates to any other one, the equations (4.3) must be unaltered in the form by the transformation

$$(4.8) \quad \bar{x}^\lambda = \bar{x}^\lambda(x^0, x^1, \dots, x^n),$$

where $\bar{x}^\lambda(x)$ are homogeneous functions of degree one with respect to their arguments.

By these transformations, the functions $II_{\mu\nu}^\lambda$ are transformed into $\bar{II}_{\mu\nu}^\lambda$ following the formulae of the same form as (1.8),

$$(4.9) \quad \bar{II}_{\mu\nu}^\lambda = \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial \bar{x}^\mu} \frac{\partial x^\tau}{\partial \bar{x}^\nu} II_{\beta\tau}^\alpha + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \right).$$

The conditions (4.5), (4.6) and (4.7) are invariant by these transformations.

As we may multiply homogeneous projective coordinates by any factor, the equations (4.3) must be invariant also by the transformation

$$(4.10) \quad 'Z^\lambda = \sigma Z^\lambda \quad \text{or} \quad 'x^\lambda = \sigma x^\lambda,$$

where the factor $\sigma(x)$ is a homogeneous function of degree zero with respect to the x^λ .

It will be easily seen that, by this transformation, the functions $II_{\mu\nu}^\lambda$ are transformed into $'II_{\mu\nu}^\lambda$ following the formulae

$$(4.11) \quad 'II_{\mu\nu}^\lambda = II_{\mu\nu}^\lambda + \delta_\mu^\lambda \sigma_\nu + \delta_\nu^\lambda \sigma_\mu + \sigma_{\mu\nu} x^\lambda,$$

where

$$(4.12) \quad \sigma_\nu = \frac{\partial \log \sigma}{\partial x^\nu} \quad \text{and} \quad \sigma_{\mu\nu} = \frac{\partial \sigma_\mu}{\partial x^\nu} - \sigma_\nu II_{\mu\nu}^\lambda - \sigma_\mu \sigma_\nu.$$

The conditions (4.5), (4.6) and (4.7) are also invariant under such transformation of coordinates.

As the transformation laws (4.9) and (4.11) of $II_{\mu\nu}^\lambda$ are transitive, we may define, as in Paragraph 1, an invariant whose components in the coordinate system (x^α) are $II_{\mu\nu}^\lambda$ and whose components $\bar{II}_{\mu\nu}^\lambda$ and $'II_{\mu\nu}^\lambda$ in other coordinate systems (\bar{x}^α) and (x^α) are given by (4.9) and (4.11) respectively. This invariant may be called the projective connection.

Thus, a generalized projective geometry⁵⁾ may be defined as the theory of an arbitrary projective connection, whose components are functions of $n+1$ curvilinear homogeneous coordinates and satisfy the conditions (4.5) and (4.6). If the components of a projective connection satisfy the conditions (4.5), (4.6) and (4.7), the theory of this projective connection is flat projective geometry.

It is to be noticed that, if the components satisfy (4.5), (4.6) and (4.7), these components are homogeneous functions of degree -1 with respect to x^α . In fact, we have, from (4.7)

5) J. A. Schouten and J. Haantjes: Zur allgemeinen projektiven Differentialgeometrie, *Comp. Math.*, **3** (1936), 1-51.

$$\mathcal{Q}^{\lambda}_{\mu\nu\omega}x^{\omega} = \left(\frac{\partial \Pi^{\lambda}_{\mu\nu}}{\partial x^{\omega}} - \frac{\partial \Pi^{\lambda}_{\mu\omega}}{\partial x^{\nu}} + \Pi^{\alpha}_{\mu\nu}\Pi^{\lambda}_{\alpha\omega} - \Pi^{\alpha}_{\mu\omega}\Pi^{\lambda}_{\alpha\nu} \right) x^{\omega} = 0,$$

from which

$$\frac{\partial \Pi^{\lambda}_{\mu\nu}}{\partial x^{\omega}} x^{\omega} = -\Pi^{\lambda}_{\mu\nu}$$

§ 5. In this Paragraph; we shall study the relations between two components of the projective connection $\Gamma^{\lambda}_{\mu\nu}$ and $\Pi^{\lambda}_{\mu\nu}$, one introduced in § 1 and the other in the last Paragraph.

The non-homogeneous curvilinear coordinates ξ^i and the homogeneous curvilinear coordinates x^{λ} being both coordinate systems for an n -dimensional projective space P_n , we must have the equations of the form

$$(5.1) \quad \begin{cases} \xi^0 = \log \rho(x^{\alpha}), \\ \xi^i = \xi^i(x^{\alpha}) \end{cases}$$

between these coordinates where the functions $\xi^0(x^{\alpha})$ and $\xi^i(x^{\alpha})$ are homogeneous of degree one and zero respectively with respect to x^{α} . Thus, if we put

$$\frac{\partial \log \rho}{\partial x^{\lambda}} = \rho_{\lambda} \quad \text{and} \quad E^{\xi^i}_{\lambda} = \frac{\partial \xi^i}{\partial x^{\lambda}},$$

we have

$$(5.2) \quad \rho_{\lambda} x^{\lambda} = 1 \quad \text{and} \quad E^{\xi^i}_{\lambda} x^{\lambda} = 0.$$

As we have

$$(5.3) \quad \frac{\partial Z}{\partial x^{\mu}} = \rho_{\mu} Z + E^{\xi^j}_{\mu} \frac{\partial Z}{\partial \xi^j},$$

$$(5.4) \quad \begin{aligned} \frac{\partial^2 Z}{\partial x^{\mu} \partial x^{\nu}} &= \frac{\partial \rho_{\mu}}{\partial x^{\nu}} Z + \rho_{\mu} \rho_{\nu} Z + E^{\xi^j}_{\mu} \rho_{\nu} \frac{\partial Z}{\partial \xi^j} + E^{\xi^j}_{\nu} \rho_{\mu} \frac{\partial Z}{\partial \xi^j} \\ &+ E^{\xi^j}_{\mu} E^{\xi^k}_{\nu} \frac{\partial^2 Z}{\partial \xi^j \partial \xi^k} + \frac{\partial E^{\xi^j}_{\mu}}{\partial x^{\nu}} \frac{\partial Z}{\partial \xi^j}, \end{aligned}$$

substituting the fundamental equations (1.3) and (4.3) in (5.4), we have

$$\begin{aligned} &\rho_{\lambda} \Pi^{\lambda}_{\mu\nu} Z + E^{\xi^i}_{\lambda} \Pi^{\lambda}_{\mu\nu} \frac{\partial Z}{\partial \xi^i} \\ &= \frac{\partial \rho_{\mu}}{\partial x^{\nu}} Z + \rho_{\mu} \rho_{\nu} Z + E^{\xi^j}_{\mu} \rho_{\nu} \frac{\partial Z}{\partial \xi^j} + E^{\xi^j}_{\nu} \rho_{\mu} \frac{\partial Z}{\partial \xi^j} + E^{\xi^j}_{\mu} E^{\xi^k}_{\nu} \Gamma^0_{jk} Z \\ &+ E^{\xi^j}_{\mu} E^{\xi^k}_{\nu} \Gamma^i_{jk} \frac{\partial Z}{\partial \xi^i} + \frac{\partial E^{\xi^j}_{\mu}}{\partial x^{\nu}} \frac{\partial Z}{\partial \xi^j} \end{aligned}$$

in virtue of the equation (3.5).

The $n+1$ points Z and $\frac{\partial Z}{\partial \xi^i}$ being independent, we have from the above

equation

$$(5.5) \quad E_{,\mu}^j E_{,\nu}^k \Gamma_{jk}^0 = - \left(\frac{\partial \rho_{\mu}}{\partial x^{\nu}} - \rho_{\lambda} \Pi_{\mu\nu}^{\lambda} - \rho_{\mu} \rho_{\nu} \right),$$

and

$$(5.6) \quad E_{,\mu}^j E_{,\nu}^k \Gamma_{jk}^i = - \left(\frac{\partial E_{,\mu}^i}{\partial x^{\nu}} - E_{,\lambda}^i \Pi_{\mu\nu}^{\lambda} + E_{,\mu}^i \rho_{\nu} + E_{,\nu}^i \rho_{\mu} \right).$$

If we denote by

$$\begin{pmatrix} x^{\lambda} \\ E_i^{\lambda} \end{pmatrix}$$

the inverse matrix of

$$\begin{pmatrix} \rho_{\lambda} \\ E_{,\lambda}^i \end{pmatrix}$$

we can solve the equations (5.5) and (5.6) with respect to Γ_{jk}^0 and Γ_{jk}^i respectively, say,

$$(5.7) \quad \Gamma_{jk}^0 = - E_j^{\mu} E_k^{\nu} \left(\frac{\partial \rho_{\mu}}{\partial x^{\nu}} - \rho_{\lambda} \Pi_{\mu\nu}^{\lambda} \right),$$

$$(5.8) \quad \Gamma_{jk}^i = - E_j^{\mu} E_k^{\nu} \left(\frac{\partial E_{,\mu}^i}{\partial x^{\nu}} - E_{,\lambda}^i \Pi_{\mu\nu}^{\lambda} \right).$$

§ 6. In this last Paragraph, we shall investigate the differential equations of paths in the case where we are using the curvilinear homogeneous coordinates. The equation of paths being

$$Z^{\lambda} = A^{\lambda} u(r) + B^{\lambda} v(r),$$

we have the differential equations

$$\frac{d^2 Z^{\lambda}}{dr^2} = \alpha \frac{dZ^{\lambda}}{dr} + \beta Z^{\lambda},$$

from which, we find

$$\frac{d^2 x^{\lambda}}{dr^2} + \Pi_{\mu\nu}^{\lambda} \frac{dx^{\mu}}{dr} \frac{dx^{\nu}}{dr} = \alpha \frac{dx^{\lambda}}{dr} + \beta x^{\lambda}$$

by virtue of the fundamental equations.

As we have already stated, these equations give in an $(n+1)$ -dimensional representation, a curve whose osculating plane passes always by the origin and hence a curve on a plane passing through the origin.

Thus the transformation (4.11) may be called the subprojective transformation⁶⁾ of the affine connection.

6) K. Yano: Subprojective transformations, subprojective spaces and subprojective collineations. Proc., 20 (1944), 701-705.