57. A note on generalized convex functions.

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§ 1. We are concerned with real finite functions $f(x)$ defined on a closed interval $a\leqq x\leqq b$. E. F. Beckenbach[1] has given a generalization of the notion of a convex function as follows.

Let $F(x;\alpha,\beta)$ be a two-parameter family of real finite functions defined on $a\leqq x\leqq b$ and satisfying the follwing conditions:

(1) each $F(x;\alpha,\beta)$ is a continuous function of x;

(2) there is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a\leqq x_1<x_2\leqq b$, takes on arbitrary values y_1, y_2.

The members of the family $F(x;\alpha,\beta)$ are denoted simply by $F(x)$, individual members being distinguished by subscripts. In particular, $F_{ij}(x)$ denotes the member satisfying $F_{ij}(x_i)=f(x_i)$, $F_{ij}(x_j)=f(x_j)$, $(a\leqq x_i<x_j\leqq b)$.

We call *a function $f(x)$ to be convex in Beckenbach's sense if*

$$f(x)\leqq F_{12}(x)$$

for all x_1, x_2, x, with $a\leqq x_1<x<x_2\leqq b$.

Now let the family $F(x;\alpha,\beta)$ satisfy the following condition (3) in addition to (1) and (2):

(3) let $F(x)$, $F'(x)$ be the members of the family passing through arbitrary points (x_1, y_1), (x_2, y_2); (x_1, y_1'), (x_2, y_2') respectively, then, the member $F_\lambda(x)(\lambda>0)$ which passes through $(x_1, \lambda y_1)$, $(x_2, \lambda y_2)$ is not less than $\lambda F(x)$ for $a\leqq x_1<x<x_2\leqq b$, and the member passing through (x_1, y_1+y_1'), (x_2, y_2+y_2') is not less than $F(x)+F'(x)$ for $a\leqq x_1<x<x_2\leqq b$.

Definition. A function $f(x)$ is called a generalized convex function if the family $F(x;\alpha,\beta)$ satisfies the condition (1), (2), and (3), and $f(x)\leqq F_{12}(x)$ for all x_1, x_2, x, with $a\leqq x_1<x<x_2\leqq b$.

For instance, (a) when $F(x;\alpha,\beta)\equiv\alpha x+\beta$, then $F(x;\alpha,\beta)$ satisfies the conditions (1), (2), and (3) and therefore the convex function in the usual sense is a generalized convex function, (b) when $F(x;\alpha,\beta)\equiv\alpha\sin\rho x+\beta\cos\rho x$ where ρ is a constant and $b-a<\frac{\pi}{\rho}$, then $F(x;\alpha,\beta)$ satisfies (1), (2), and (3) and

1) E. F. Beckenbach; Generalized convex functions, Bull. Amer. Math. Soc. Vol. 43 (1937), 363-371.

therefore the Lindelöf-Phragmén's function[2] $h(x)$ is a generalized convex function.

The purpose of the present paper is to show that the set of all generalized convex functions is a semi-vector lattice[3] and the set of all convex functions in Beckenbach's sense is not always a semi-vector lattice.

§ 2. It was shown by E. F. Beckenbach[4] that the family $F(x;\alpha,\beta)$ satisfying the conditions (1), (2) and the convex functions in Beckenbach's sense have several properties analogous to those of the family $F(x;\alpha,\beta)\equiv\alpha x+\beta$ and the convex functions in the usual sense. These properties will be used to establish the following theorem.

Theorem 1. The set of all generalized convex functions is a semi-vector lattice.

Proof. Let F denote the set of all generalized convex functions and let $f(x)\in F$, $g(x)\in F$. Then there exist two members $F_{12}(x)$, $F'_{12}(x)$ of the family $F(x;\alpha,\beta)$ such that

$$f(x)\leqq F_{12}(x),\ g(x)\leqq F'_{12}(x)$$

for all x_1, x_2, x, with $a\leqq x_1<x<x_2\leqq b$.
Then we have

$$f(x)+g(x)\leqq F_{12}(x)+F'_{12}(x)$$

for all x_1, x_2, x, with $a\leqq x_1<x<x_2\leqq b$.
On the other hand, it follows from (3) that there is a unique member $F''_{12}(x)$ of the family $F(x;\alpha,\beta)$ such that $F_{12}(x)+F'_{12}(x)\leqq F''_{12}(x)$ for all x_1, x_2, x, with $a\leqq x_1<x<x_2\leqq b$ and $F''_{12}(x_1)=f(x_1)+g(x_1)$, $F''_{12}(x_2)=f(x_2)+g(x_2)$.
Therefore $$f(x)+g(x)\in F.$$
From (3), it is easy to see that $\lambda f(x)\in F$ for $\lambda>0$.
Conequently, the set F is a semi-linear space.

In order to prove that the set F forms a lattice, it is sufficient to show that an arbitrary bounded subset E of elements of the set F has the supremum $U(x)\equiv \underset{f\in E}{l.u.b.}\, f(x)$.
Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be arbitrary two points on the curve $y=U(x)$ such that $x_1<x_2$, and let $P_0(x_0, y_0)$ be any point on the arc of $y=U(x)$ between the points P_1, P_2.
We need only consider the following two cases, on account of being able to prove

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2) 辻正次；解析雜論 (共立社), 20 頁.

3) G. Birkhoff; Lattice Theory, P. 107, there it is shown that the set of all convex functions in the usual sense forms a semi-vector lattice.

4) E. F. Beckenbach; loc. cit.

similarly in the other cases.

Case 1. All the points P_0, P_1, P_2 lie on the curves of elements of E. Let $P'_2(x_2, y'_2)$ be the point of intersection of an element of E passing through the point P_0 with the line $x=x_2$. Then $y'_2 \leqq y_2$.

Therefore we have

$$F'_{02}(x) \leqq F_{02}(x) \text{ for all } x \text{ with } x_0 \leqq x \leqq x_2,$$ [5]

where $F'_{02}(x)$, $F_{02}(x)$ donote the members of the family $F(x;\alpha, \beta)$ which pass through the points P_0, P'_2; P_0, P_2 respectively.

On the other hand, since $F_{02}(x_1) < y_1$[6] and $F_{12}(x_2) = F_{02}(x_2)$, we have

$$F_{02}(x) < F_{12}(x) \text{ for all } x \text{ with } x_1 \leqq x \leqq x_2,$$

where $F_{12}(x)$ denotes the member of the family $F(x;\alpha, \beta)$ passing through the points P_1, P.

Therefore we have

$$F'_{02}(x) < F_{12}(x) \text{ for all } x \text{ with } x_0 \leqq x \leqq x_2.$$

Consequently we get

$$U(x_0) = F'_{02}(x_0) < F_{12}(x_0).$$

Since P_0 is an arbitrary point on the arc of $y=U(x)$ between the points P_1, P_2, we have

$$U(x) < F_{12}(x) \text{ for all } x \text{ with } x_1 \leqq x \leqq x_2.$$

Hence we have $U(x) \in F$.

Case 2. Both of the points P_1, P_2 do not lie on the curves of elements of E.

Let $\{P_{1n}\}$, $\{P_{2n}\}$ be monotone increasing sequences of points of intersection of the lines $x=x_1$, $x=x_2$ with elements of E respectively such that $P_{1n} \to P_1$, $P_{2n} \to P_2$, as $n \to \infty$.

Then we have

$$F_{12n}(x) \to F_{12}(x) \text{ as } n \to \infty,$$ [7]

where $F_{12n}(x)$ denote the members of $F(x;\alpha, \beta)$ passing through the points P_{1n}, P_{2n}.

Therefore, in virtue of the monotone increase of $\{P_{1n}\}$, $\{P_{2n}\}$, and by the same argument as in Case 1, we have

$$U(x) < F_{12}(x) \text{ for all } x \text{ with } x_1 \leqq x \leqq x_2.$$

Therefore in this case we have $U(x) \in F$.

From the proofs of the above two cases and the fact that F is a semi-linear space, we get the theorem.

Concerning the condition (3), we have the following theorem.

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5), 6) 7) E. F. Beckenbach; loc. cit.

Theorem 2. The set of all convex functions in Beckenbach's sense is not always a semi-vector lattice.

Proof. For the proof, we consider an example which is not a semi-vector lattice.

Let $F(x;\alpha,\beta)\equiv -x^2+\alpha x+\beta$ defined on the closed interval $-1\leqq x\leqq 1$, then it is easy to see that the family $F(x;\alpha,\beta)$ satisfies (1), (2), but does not satisfy (3).

It is evident that the member $F_{12}(x)$ of the family $F(x;\alpha,\beta)$ passing through the points $P_1(-1, 0)$, $P_2(1, 0)$ is $-x^2+1$ and $F_{12}(x)$ is itself a convex function in Beckenbach's sense.

Since $\lambda F_{12}(x)$ where $\lambda>1$ is greater than $F_{12}(x)$ which is the member of the family $F(x;\alpha,\beta)$ passing through P_1, P_2, so that $\lambda F_{12}(x)$ is not a convex function in Beckendach's sense.

Therefore the set of all convex functions in Bekenbach's sense corresponding to $F(x;\alpha,\beta)\equiv -x^2+\alpha x+\beta$ is not a semi-linear space.

Thus the proof of the theorem is completed.

Remark. From the proof of Theorem 1, we see that the set of all convex functions-in Beckenbach's sense is a conditionally complete lattice.