48. *On a Regular Function, whose Real Part is Positive in a Unit Circle.*

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1. Carathéodory's theory[1] of positive harmonic functions in a unit circle atracted interests of many mathematicians[1] and several proofs were given and the results were completed and now the main results stand in the following theorems. In this peper, I will give a simple proof, where the proof of Theorem 1(I) is suggested by Szasz's paper[1] and the proof of Theorem 1(II) is the same as Schur's proof[1] essentially, but in a modified form.

Teeorem 1. *Let* $f(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nz^n$ (a_0=*real*) *be regular in* $|z|<1$. *Then* (I)(*Carathéodory*[1]-*Toeplitz*).[2] $\Re f(z)\geqq 0$ *in* $|z|<1$, *when and only when the Hermitian forms* $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu(a_{-\nu}=\bar{a}_\nu)$[3] *are non-negative for* $n=0, 1, 2,\ldots$ *If* all $H_n(x)$ are non-negative and $H_0(x),\ldots, H_{k-1}(x)$ *are positive definite and* $H_k(x)$ *is positive semi-definite, then* $f(z)$ *is of the form:*

$$f(z)=\sum_{\nu=1}^{k}\frac{r_\nu}{2}\cdot\frac{1+\epsilon_\nu z}{1-\epsilon_\nu z},\ (r_\nu>0,\ |\epsilon_\nu|=1,\ \epsilon_i\neq\epsilon_j(i\neq j)),\quad (1)$$

where k *is the rank of the infinite Hermitian matrix* H:

$$H=\begin{pmatrix} a_0, & a_1, & a_1,\ldots \\ \bar{a}_1, & a_0, & a_1,\ldots \\ \bar{a}_2, & \bar{a}_1, & a_0,\ldots \\ \ldots\ldots\ldots \end{pmatrix}.$$

(II) (*I. Schur*).[1] *If we put*

1) C. Carathéodory: Über die Variabilitatsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. Rendiconti del circolo mat. Palermo. **32** (1911).

2) O. Toeplitz: Über die Fouriersche Entwicklung positiver Funktionen. Rendiconti del circolo mat. Palermo. **32** (1911). E. Fischer: Über das Carathéodorysche Problem. Rendiconti del circolo mat. Palermo. **32** (1911). I. Schur: Über potenzreihen, die in Innern des Einheitskreises beschränkt sind. Crelle. **147** (1917). O. Szasz: Über harmonischen Funktionen und L Formen. Math. Zeits. **1** (1918). G. Szegö Über Funktionen mit positiver Realteil. Math. Ann. **99** (1928). F. Riesz: Über ein Problem des Herrn Carathéodory. Crelle **146** (1916).

3) In this paper, $\bar{a}$ means the conjugate complex of a.

$$\delta_n=\delta(a_0, a_1,\dots, a_n)=\begin{vmatrix} a_0, a_1, & a_1, & \dots, a_n \\ \bar{a}_1, a_0, & a_1, & \dots, a_{n-1} \\ \dots\dots\dots & & \\ \bar{a}_n, \bar{a}_{n-1}, & \bar{a}_{n-2}, & \dots, a_0 \end{vmatrix}, \qquad (2)$$

then $\Re f(z)\geqq 0$ *in* $|z|<1$, *when and only when* (i) $\delta_n>0$ *for all n or* (ii) $\delta_0>0, \delta_1>0,\dots, \delta_{k-1}>0, \delta_k=\delta_{k+1}=\dots=0$ *for some k. This case occurs only when* $f(z)$ *is of the form* (1).

Theorem 2. (Carathéodory).[4] *Let* $a_0, a_1,\dots,a_n$ $(a_0=real)$ *be* $n+1$ *complex numbers, such that* $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ *is non-negative. Then there exists a regular function* $f(z)$ *in* $|z|<1$, *such that* $\Re f(z)\geqq 0$ *in* $|z|<1$ and

$$f(z)=\frac{a_0}{2}+a_1z+\dots+a_nz^n \ (mod.\ z^{n+1}).$$

If $H_n(x)$ *is positive semi-definite, then such* $f(z)$ *is unique and is of the form* (1), *where* $k\leqq n$.

For the proof, we use the followong theorems.

Theorem A. (Fejer).[5] *Let* $\tau(\varphi)=\lambda_0+\sum_{\nu=1}^n(\lambda_\nu\cos\nu\varphi+\mu_\nu\sin\nu\varphi)\geqq 0$ *in* $[0, 2\pi]$. *Then* $\tau(\varphi)$ *can be expressed in the form:*

$$\tau(\varphi)=|\gamma_0+\gamma_1e^{i\varphi}+\dots+\gamma_ne^{in\varphi}|^2.$$

Theorem B. (I. Schur).[6] *Let* $A=\sum_1^n a_{\nu\mu}x_\nu\bar{x}_\mu$ *be a Hermitian form, such that*

$$a'\leqq A\leqq a \ for \ |x_1|^2+|x_2|^2+\dots+|x_n|^2=1$$

and $B=\sum_1^n b_{\nu\mu}x_\nu\bar{x}_\mu$ *be a non-negative Hermitian form,*

$$b'=Min.\ (b_{11}, b_{22},\dots, b_{nn}),\ b=Max.\ (b_{11}, b_{22}, \dots b_{nn}).$$

Then

$$a'b'\leqq\sum_1^n a_{\nu\mu}b_{\nu\mu}x_\nu\bar{x}_\mu\leqq ab$$

$$for \quad |x_1|^2+|x_1|^2+\dots+|x_n|^2=1.$$

4) C. Carathéodory. l.c. (1). I. Schur: Über einen Satz von C. Carathéodory. Berliner Ber. 1912. G. Frobenius: Ableitung eines Satzes von Carathéodory aus einer Formel von Kronecker. Berliner Ber. 1912.

5) L. Féjér: Über trigonometrische Polynome. Crelle **146** (1916).

6) I. Schur: Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veranderlichen. Crelle **140** (1911). O. Szasz. l.c. (2).

2. *Proof of Theorem* 1(I).

(i) Let $\Re f(z)\geqq 0$ in $|z|<1$, then by Herglotz's theorem, $f(z)$ can be expressed by

$$f(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nz^n=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\chi(\varphi), \qquad (3)$$

where $\chi(\varphi)$ is a non-decreasing function of φ, so that

$$a_n=\frac{1}{\pi}\int_0^{2\pi}e^{-in\varphi}d\chi(\varphi)(n=0,\ 1,\ 2,\ldots). \qquad (4)$$

Hence

$$H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu=\frac{1}{\pi}\int_0^{2\pi}|x_0+x_1e^{i\varphi}+\ldots+x_ne^{in\varphi}|^2\,d\chi\geqq 0. \qquad (5)$$

$$(n=0,1,2,\ldots)$$

(ii) Next we will prove that $\Re f(z)\geqq 0$ in $|z|<1$, if all $H_n(x)$ are non-negative. Since for $|z|<\rho<1$,

$$f(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nz^n=\frac{1}{2\pi}\int_0^{2\pi}\Re f(\rho e^{i\varphi})\frac{\rho e^{i\varphi}+z}{\rho e^{i\varphi}-z}d\varphi, \qquad (6)$$

we have

$$a_n\rho^n=\frac{1}{\pi}\int_0^{2\pi}\Re f(\rho e^{i\varphi})e^{-in\varphi}d\varphi(n=0,\ 1,\ 2,\ldots), \qquad (7)$$

so that

$$H_n^{(\rho)}(x)=\sum_0^n a_{\mu-\nu}\rho^{|\mu-\nu|}x_\nu\bar{x}_\mu=$$

$$\frac{1}{\pi}\int_0^{2\pi}\Re f(\rho e^{i\varphi})\,|\,x_0+x_1e^{i\varphi}+\ldots+x_ne^{in\varphi}\,|^2d\varphi. \qquad (8)$$

Let

$$g(z)=\frac{1}{2}+\sum_{n=1}^{\infty}z^n=\frac{1+z}{2(1-z)},\ \text{then } \Re g(z)=\frac{1-|z|^2}{2|1-z|^2}>0 \text{ in } |z|<1.$$

Hence by (8),

$$B=\sum_0^n\rho^{|\mu-\nu|}\,x_\nu\bar{x}_\mu=\frac{1}{\pi}\int_0^{2\pi}\Re g(\rho e^{i\varphi})\,|\,x_0+x_1e^{i\varphi}+\ldots+x_ne^{in\varphi}|^2d\varphi\geqq 0. \quad (9)$$

We put for $|x_0|^2+|^2+\ldots+|x_n|^2=1$,

$$g_n=\text{Min. } H_n(x),\ G_n=\text{Max. } H_n(x),$$

$$g_n^{(\rho)}=\text{Min. } H_n^{(\rho)}(x),\ G_n^{(\rho)}=\text{Max. } H_n^{(\rho)}(x). \qquad (10)$$

Since $g_0^{(\rho)}\geqq g_1^{(\rho)}\geqq\ldots\geqq g_n^{(\rho)},\ldots G_0^{(\rho)}\leqq G_1^{(\rho)}\leqq\ldots\leqq G_n^{(\rho)}$, let

$$\lim_{n\to\infty}g_n^{(\rho)}=g^{(\rho)},\ \lim_{n\to\infty}G_n^{(\rho)}=G^{(\rho)}. \qquad (11)$$

We apply Theorem B on $A=H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$, $B=\sum_0^n \rho^{|\mu-\nu|}x_\nu\bar{x}_\mu$, then since $H_n(x)\geqq 0$ and $b_{11}=b_{22}=\dots=b_{nn}=1$, we have $0\leqq g_n\leqq g_n^{(\rho)}\leqq G_n^{(\rho)}\leqq G_n$, so that

$$0\leqq g^{(\rho)}\leqq g_n^{(\rho)}\leqq G_n^{(\rho)}\leqq G^{(\rho)}. \tag{12}$$

Let $\rho e^{i\varphi_0}$ be any point on $|z|=\rho$, then $|\Re f(\rho e^{i\varphi})-\Re f(\rho e^{i\varphi_0})|<\epsilon$ for $|\varphi-\varphi_0|\leqq\delta$. We define a positive continuous function $g(\varphi)$ in $[0,2\pi]$ by the following conditions: (i) $\int_0^{2\pi} g(\varphi)d\varphi=2\pi$, (ii) $g(\varphi)=\text{const.}=M(>0)$ in $|\varphi-\varphi_0|\leqq\delta$ and is a linear function in $[\varphi_0-\delta-\delta', \varphi_0-\delta]$ and $[\varphi_0+\delta, \varphi_0+\delta+\delta']$, such that $g(\varphi_0-\delta-\delta')=\eta$, $g(\varphi_0-\delta)=M$, $g(\varphi_0+\delta)=M$, $g(\varphi_0+\delta+\delta')=\eta$ $(\eta>0)$ and $g(\varphi)=\text{const.}=\eta$ in the remaining part of $[0, 2\pi]$, where we take M so large and δ', η so small, that

$$\int_J g(\varphi)d\varphi<\epsilon, \text{ so that} \int_{\varphi_0-\delta}^{\varphi_0+\delta} g(\varphi)d\varphi=2\pi-0(\epsilon), \tag{13}$$

where J is the complementary set of $|\varphi-\varphi_0|\leqq\delta$ in $[0, 2\pi]$.

Now we approximate $g(\varphi)$ by a trigonometrical polynomial $\tau(\varphi)$ of order n, such that $|g(\varphi)-\tau(\varphi)|<\epsilon_1(<\eta)$ in $[0, 2\pi]$, then $\tau(\varphi)>0$ in $[0, 2\pi]$. Hence by Theorem A, $\tau(\varphi)=|x_0+x_1e^{i\varphi}+\dots+x_ne^{in\varphi}|^2$ with suitable $x_0, x_1,\dots, x_n$. Then

$$2\pi(|x_0|^2+|x_1|^2+\dots+|x_n|^2)=\int_0^{2\pi}\tau(\varphi)d\varphi=\int_0^{2\pi}g(\varphi)d\varphi+0(\epsilon_1)=2\pi+0(\epsilon_1),$$

so that $|x_0|^2+|x_1|^2+\dots+|x_n|^2=1+0(\epsilon_1)$.

From (8), (12), (13),

$$\begin{aligned}0\leqq g^{(\rho)}\cdot(|x_0|^2+|x_1|^2+\dots+|x_n|^2)&\leqq H_n^{(\rho)}(x)=\frac{1}{\pi}\int_0^{2\pi}\Re f(\rho e^{i\varphi})\tau(\varphi)d\varphi\\ &=\frac{1}{\pi}\int_0^{2\pi}\Re f(\rho e^{i\varphi})g(\varphi)d\varphi+0(\epsilon_1)=\frac{1}{\pi}\int_{\varphi_0-\delta}^{\varphi_0+\delta}\Re f(\rho e^{i\varphi})g(\varphi)d\varphi\\ &+\frac{1}{\pi}\int_J\Re f(\rho e^{i\varphi})g(\varphi)d\varphi+0(\epsilon_1)=\frac{\Re f(\rho e^{i\varphi_0})}{\pi}\int_{\varphi_0-\delta}^{\varphi_0+\delta}g(\varphi)d\varphi+0(\epsilon)\\ &+0(1)\int_J g(\varphi)d\varphi+0(\epsilon_1)=2\Re f(\rho e^{i\varphi_0})+0(\epsilon)+0(\epsilon_1).\end{aligned}$$

Making $\epsilon\to 0$, $\epsilon_1\to 0$ we have

$$0\leqq g^{(\rho)}\leqq 2\Re f(\rho e^{i\varphi_0}). \tag{14}$$

Hence $\Re f(z)\geqq 0$ in $|z|<1$, q.e.d.

(iii) Suppose that all $H_n(x)$ are non-negative and $H_0(x), H_1(x),\dots, H_{k-1}(x)$ are positive definite and $H_k(x)$ is positive semi-definite. Then there exists $x_0', x_1',\dots, x_k'$ $(|x_0'|^2+|x_1'|^2+\dots+|x_k'|^2=1)$, such that $H_k(x')=0$ Since by (ii) $\Re f(z)\geqq 0$ in $|z|<1$, we have by (5),

$$\int_0^{2\pi} |x_0' + x_1' e^{i\varphi} + \dots + x_k' e^{ik\varphi}|^2 d\chi = 0. \tag{15}$$

If $\chi(\varphi)$ is increasing at $\varphi=\varphi_0$ and $x_0' + x_1' e^{i\varphi_0} + \dots + x_k' e^{ik\varphi_0} \neq 0$, then $|x_0' + x_1' e^{i\varphi} + \dots + x_k' e^{ik\varphi}| \geqq \eta > 0$ for $|\varphi-\varphi_0| \leqq \delta$, so that $\int_0^{2\pi} |x_0' + x_1' e^{i\varphi} + \dots + x_k' e^{ik\varphi}|^2 d\varphi > 0$, which contradicts to (15). Hence if $\chi(\varphi)$ is increasing at $\varphi=\varphi_0$, then $x_0' + x_1' e^{i\varphi_0} + \dots + x_k' e^{ik\varphi_0} = 0$. Since $x_0' + x' z + \dots + x_k' z^k = 0$ has at most k roots on $|z|=1$, $\chi(\varphi)$ is increasing at $\varphi_1, \dots, \varphi_j (j \leqq k)$ and is constant outside φ_ν, so that by (5),

$$H_n(x) = \sum_{\nu=1}^{j} r_\nu |x_0 + x_1 e^{i\varphi_\nu} + \dots + x_n e^{in\varphi_\nu}|^2 \qquad \left(r_\nu = \frac{d\chi(\varphi_\nu)}{\pi} > 0\right),$$

$$(n=0, 1, 2, \dots).$$

If $j \leqq k-1$, then a system of linear equations:

$$x_0 + x_1 e^{i\varphi_\nu} + \dots + x_{k-1} e^{i(k-1)\varphi_\nu} = 0 \quad (\nu=1, 2, \dots, j)$$

has a solution $x_0'', x_1'', \dots, x_{k-1}''$, such that $|x_0''|^2 + |x_1''|^2 + \dots + |x_{k-1}''|^2 = 1$. Then $H_{k-1}(x'') = 0$, which contradicts the hypothesis. Hence $j=k$, so that by (3),

$$f(z) = \sum_{\nu=1}^{k} \frac{r_\nu}{2} \cdot \frac{1+\epsilon_\nu z}{1-\epsilon_\nu z} \quad (\epsilon_\nu = e^{-i\varphi_\nu}), \tag{16}$$

hence

$$a_n = r_1 \epsilon_1^n + \dots + r_k \epsilon_k^n,$$

$$H_n(x) = \sum_{\nu=1}^{k} r_\nu |x_0 + x_1 \epsilon_\nu^{-1} + \dots + x_n \epsilon_\nu^{-n}|^2. \tag{17}$$

$$(n=0, 1, 2, \dots)$$

From (17), we see easily that k is the rank of H.

Conversely, if $f(z)$ is of the form (1), then $H_0(x), \dots, H_{k-1}(x)$ are positive definite and $H_k(x)$ is positive semi-definite.

3. A Lemma to the proof of Theorem 1 (II).

Let $f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n$ be regular for $|z| < 1$ and suppose that $a_0 > 0$, we define $f_1(z)$ by the following relations:

$$\varphi(z) = \frac{1-f(z)}{1+f(z)} = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots, \quad \alpha_0 = \frac{2-a_0}{2+a_0}, \ (|\alpha_0| < 1),$$

$$\varphi_1(z) = \frac{\epsilon}{z} \cdot \frac{\varphi(z) - \alpha_0}{1 - \alpha_0 \varphi(z)}, \ (|\epsilon| = 1),$$

$$f_1(z) = \frac{1-\varphi_1(z)}{1+\varphi_1(z)} = \frac{(\alpha_0 + \epsilon\alpha_1) + (\alpha_1 + \epsilon\alpha_2) z + \dots + (\alpha_n + \epsilon\alpha_{n+1}) z^n + \dots}{(\alpha_0 - \epsilon\alpha_1) + (\alpha_1 - \epsilon\alpha_2) z + \dots + (\alpha_n - \epsilon\alpha_{n+1}) z^n + \dots}$$

$$=\frac{c_0}{2}+c_1z+c_2z^2+\dots,\quad c_0=2\frac{a_0+\epsilon a_1}{a_0-\epsilon a_1}, \tag{18}$$

where we determine $\epsilon(|\epsilon|=1)$, such that $a_0-\epsilon a_1\neq 0$ and c_0 is real. That this is always possible is seen as follows. If $a_1=0$, then we take $\epsilon=1$ and if $a_1\neq 0$, we take as ϵ one of solutions of $\epsilon^2=\frac{\bar{a}_1}{a_1}$, such that $a_0-\epsilon a_1\neq 0$, which is possible, since if $a_0-\epsilon a_1=0$, $a_0+\epsilon a_1=0$, then we would have $a_0=0$, $a_1=0$, which contradicts to $a_0>0$. c_0 is real since $\epsilon^2=\frac{\bar{a}_1}{a_1}$. Then we have

Lemma. $\delta(c_0, c_1,\dots, c_\nu)=\dfrac{2^{\nu+1}a_0^\nu}{|a_0-\epsilon a_1|^{2\nu+2}}\delta(a_0, a_1,\dots, a_{\nu+1})$.

Proof. Let for any matrix A, we denote its ν-th section by A_ν, which is a matrix formed with elements of A lying in the first ν rows and first ν columns and put $|A_\nu|=\det. A_\nu$. Let

$$A=\begin{pmatrix} a_0+\epsilon a_1, & a_1+\epsilon a_2, & a_2+\epsilon a_3,\dots \\ 0, & a_0+\epsilon a_1, & a_1+\epsilon a_2,\dots \\ 0, & 0, & a_1+\epsilon a_2,\dots \\ \dots\dots\dots\dots & & \end{pmatrix},\quad \bar{A}=\begin{pmatrix} a_0+\bar{\epsilon}\bar{a}_1, & 0, & 0, & \dots \\ \bar{a}_1+\bar{\epsilon}\bar{a}_2, & a_0+\bar{\epsilon}\bar{a}_1, & 0, & \dots \\ \bar{a}_2+\bar{\epsilon}\bar{a}_3, & \bar{a}_1+\bar{\epsilon}\bar{a}_2, & a_0+\bar{\epsilon}\bar{a}_1,\dots & \\ \dots\dots\dots\dots & & & \end{pmatrix}$$

be infinite matrices and B, $\bar{B}'$, C, $\bar{C}'$ be infinite matrices similarly formed with $a_0-\epsilon a_1$, $a_1-\epsilon a_2,\dots$and $\frac{c_0}{2}$, c_1, $c_2,\dots$ respectively.

Let

$$H=C+\bar{C}'=\begin{pmatrix} c_0, & c_1, & c_2,\dots \\ \bar{c}_1, & c_0, & c_1,\dots \\ \bar{c}_2, & \bar{c}_1, & c_0,\dots \\ \dots\dots\dots & & \end{pmatrix},\qquad H_{\nu+1}=\begin{pmatrix} c_0, & c_1, & c_2, & \dots, & c_\nu \\ \bar{c}_1, & c_0, & c_1, & \dots, & c_{\nu-1} \\ \bar{c}_2, & \bar{c}_1, & c_0, & \dots, & c_{\nu-2} \\ \dots\dots\dots\dots & & & & \\ \bar{c}_\nu, & \bar{c}_{\nu-1}, & \bar{c}_{\nu-2}, & \dots, & c_0 \end{pmatrix}.$$

Then $|H_{\nu+1}|=\delta(c_0, c_1,\dots c_\nu)$. From (18), we have $\frac{A}{B}=C$, $A=CB$, $AB^{-1}=C$[7], so that $H=AB^{-1}+(\bar{B}')^{-1}\bar{A}'$, hence $\bar{B}'H\,B=\bar{B}'\,A+\bar{A}'\,B$.

Now

$$\bar{B}'A+\bar{A}'B=\begin{pmatrix} 2(a_0a_0-\bar{a}_1a_1), & 2(a_0a_1-\bar{a}_1a_2), & 2(a_0a_2-\bar{a}_1a_3),\dots \\ 2(a_0\bar{a}_1-\bar{a}_2a_1), & 2(a_0a_0-\bar{a}_2a_2), & 2(a_0a_1-\bar{a}_2a_3),\dots \\ 2(a_0\bar{a}_2-\bar{a}_3a_1), & 2(a_0\bar{a}_1-\bar{a}_3a_2), & 2(a_0a_0-\bar{a}_3a_3),\dots \\ \dots\dots\dots\dots\dots & & \end{pmatrix}. \tag{19}$$

7) I. Schur. l. c. (2)

Then

$$|(\bar{B}'A+\bar{A}'B)_{\nu+1}| = |(\bar{B}'\,HB)_{\nu+1}| = |\bar{B}'_{\nu+1}||H_{\nu+1}||B_{\nu+1}|$$
$$= |a_0-\epsilon a_1|^{2\nu+2}\,|H_{\nu+1}| = |a_0-\epsilon a_1|^{2\nu+2}\,\delta(c_0,\,c_1,\dots\,c_\nu). \quad (20)$$

We apply Sylvester's theorem on the upper left corner element a_0 of $\delta(a_0,\,a_1,\dots,\,a_{\nu+1})$, then we have $|(B'A+A'B)_{\nu+1}| = 2^{\nu+1}\,a_0^{\nu}\delta(a_0,\,a,\dots a_{\nu+1})$, so that

$$\delta(c_0,\,c_1,\dots,\,c_\nu)=\frac{2^{\nu+1}a_0^{\nu}}{|a_0-\epsilon a_1|^{2\nu+2}}\delta(a_0,\,a_1,\dots,\,a_{\nu+1}).$$

4. *Proof of Theorem* 1 (II).

If $\Re f(z)\geqq 0$ in $|z|<1$, then by Theorem 1(I), all $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ are non-negative, so that (i) $\delta_n>0$ for all n or (ii) $\delta_0>0$, $\delta_1>0,\dots$, $\delta_{k-1}>0$, $\delta_k=\delta_{k+1}=\dots=0$ for some k. Conversely, if $\delta_n>0$ for all n, then $H_n(x)$ are positive definite, so that by Theorem 1(I), $\Re f(z)\geqq 0$ in $|z|<1$. Next we will prove that $\Re f(z)\geqq 0$ in $|z|<1$ in case (ii).

First we remark that if all $\delta_n=0$, then all $a_n=0$. For, from $\delta_0=0$, we have $a_0=0$. Suppose that $a_0=a_1=\dots=a_{k-1}=0$ be proved, then $\delta(a_0,\,a_1,\dots,\,a_{2k-1})$ $=\delta(0,\,0,\dots 0,\,a_k,\dots,\,a_{2k-1})=(-1)^k|a_k|^{2k}=0$, $a_k=0$. Hence by induction, all $a_n=0$, so that $f(z)\equiv 0$, $\Re f(z)=0$ in $|z|<1$.

Suppose by induction that it is proved that $\Re f(z)\geqq 0$ in $|z|<1$, if $\delta_0>0$, $\delta_1>0,\dots$, $\delta_{k-2}>0$, $\delta_{k-1}=\dots=0$, the case $k=1$ being proved above, and let $\delta(a_0)>0$, $\delta(a_0,\,a_1)>0,\dots$, $\delta(a_0,\,a_1,\dots,\,a_{k-1})>0$, $\delta(a_0,\,a_1,\dots,\,a_k)=\dots=0$. $f_1(z)=\frac{c_0}{2}+c_1z+c_2z^2+\dots$be the function defined in the lemma, then by the lemma, we have $\delta(c_0)>0$, $\delta(c_0,\,c_1)>0,\dots$, $\delta(c_0,\,c_1,\dots,\,c_{k-2})>0$, $\delta(c_0,\,c_1,\dots,\,c_{k-1})$ $=\dots=0$, so that by induction, $\Re f_1(z)\geqq 0$ in $|z|<1$. From this we conclude easily that $\Re f(z)\geqq 0$ in $|z|<1$. Since $\Re f(z)\geqq 0$ in $|z|<1$, all $H_n(x)$ $=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ are non-negative and since $\delta(a_0)>,\dots$, $\delta(a_0,\,a,\dots a_{k-1})>0$, $\delta(a_0,\,a_1,\dots a_k)=\dots=0$, $H_0(x)$, $\dots$, $H_{k-1}(x)$ are positive definite and $H_k(x)$ is positive semi-definite, so that $f(z)$ is of the form (1).

5. *Proof of Theorem 2.*

We consider $(a_0,\,a_1,\dots,\,a_n)$ as a point in a $2n+2$-dimensional space and consider with Caratheodory a domain $\mathfrak{K}_n$:

$$\mathfrak{K}_n:\ \delta(a_0)>0,\ \delta(a_0,\,a_1)>0,\dots,\ \delta(a_0,\,a_1,\dots,\,a_n)>0.$$

Then $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ is positive definite. From this we see easily that $\mathfrak{K}_n$ is a convex domain. Its boundary consists of points:

$$\delta(a_0)>0,\ \delta(a_0,\,a_1)>0,\dots,\ \delta(a_0,\,a_1,\dots,\,x_{k-1})>0,\ \delta(a_0,\,a_1,\dots,\,a_k)=\dots=0$$

for some $k \leqq n$, since the boundary point corresponds to a positive semi-definite form $H_n(x)$. Suppose that $H_n(x)$ is non-negative. Then $(a_0, a_1, \dots, a_n)$ lies in or on the boundary of $\mathfrak{R}_n$, so that there exists in its ϵ-neighbourhood a point $(a_0', a_1', \dots, a_n')$ which lies in $\mathfrak{R}_n$, so that $\delta(a_0') > 0$, $\delta_1'(a_0', a_1') > 0, \dots$ $\delta_n' = \delta(a_0', a_1', \dots, a_n') > 0$. Let $F(z) = \delta(a_0', a_1', \dots a_n', z) = Az\bar{z} + Bz + \bar{B}\bar{z} + C$, where A, C are real.

Then $A = -\delta_{n-1}' \neq 0$. Since by Jacobi's theorem, $\begin{vmatrix} \delta_n', & B \\ \bar{B}, & \delta_n' \end{vmatrix} = \delta_{n-1}' C = -AC$, $|B|^2 - AC = \delta_n'^2 > 0$, $F(z) = 0$ is a real circle with a radius $r = \sqrt{|B|^2 - AC}$. Hence there exists a_{n+1}', such that $F(a_{n+1}') = 0$. Then $(a_0', a_1', \dots, a_{n+1}')$ belongs to the boundary of $\mathfrak{R}_{n+1}$, so that $\begin{vmatrix} a_0', & a_{n+1}' \\ \bar{a}_{n+1}', & a_0' \end{vmatrix} \geqq 0$, $|a_{n+1}'| \leqq |a_0'|$. Hence if we make $a_0' \to a_0, \dots, a_n' \to a_n$, then the corresponding a_{n+1}' are bounded, so that we can select a convergent sequence from a_{n+1}', such that $a_{n+1}' \to a_{n+1}$, then $(a_0, a_1, \dots, a_{n+1})$ belongs to the boundary of $\mathfrak{R}_{n+1}$, so that the corresponding Hermitian form $H_{n+1}(x)$ is positive semi-definite. Similarly we can find a_{n+2}, a_{n+3}, $\dots$, such that $H_{n+2}(x)$, $H_{n+3}(x), \dots$ are pisitive semi-definite, so that $|a_\nu| \leqq a_0$ $(\nu = 1, 2, \dots)$. Hence $f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n = \frac{a_0}{2} + a_1 z + \dots + a_n z^n (\text{mod. } z^{n+1})$ is regular in $|z| < 1$ and by Theorem 1(I), $\Re f(z) \geqq 0$ in $|z| < 1$.

Next we will prove that if $H_n(x)$ is postive semi-definite, such $f(x)$ is unique. Suppose that $H_0(x)$, $H_1(x), \dots, H_{k-1}(x)$ are positive definite and $H_k(x)$ $(k \leqq n)$ is positive semi-definite, then by Theorem 1(I), $f(z)$ is of the form (1), so that $a_\nu = r_1 \epsilon_1^\nu + \dots + r_k \epsilon_k^\nu (\nu = 0, 1, 2, \dots)$. Hence $\epsilon_1, \dots, \epsilon_k$ are roots of the equation:

$$F_k(x) = \begin{vmatrix} a_0, & a_1, & \dots, a_{k-1}, & a^k \\ \bar{a}_1, & a_0, & \dots, a_{k-2}, & a_{k-1} \\ \dots\dots & & & \\ \bar{a}_{k-1}, & \bar{a}_{k-2}, & \dots, a_0, & a_1 \\ 1, & x, & \dots, x^{k-1}, & x^k \end{vmatrix}$$

$$= \begin{vmatrix} r_1, & r_2, & \dots, r_k, & 0 \\ r_1\epsilon_1^{-1}, & r_2\epsilon_2^{-1} & \dots, r_k\epsilon_k^{-1}, & 0 \\ \dots\dots & & & \\ r_1\epsilon_1^{-k+1}, & r_2\epsilon_2^{-k+1}, & \dots, r_k\epsilon_k^{-k+1}, & 0 \\ 0, & 0, & \dots, 0, & 1 \end{vmatrix} \cdot \begin{vmatrix} 1, \epsilon_1, \dots, \epsilon_1^k \\ \dots\dots \\ 1, \epsilon_k, \dots, \epsilon_k^{k-1} \\ 1, x, \dots, x^k \end{vmatrix} = \delta(a_0, a_k, \dots a_{k-1}) x^k + \dots = 0, \tag{21}$$

so that $\epsilon_1, \dots, \epsilon_k$ are unique and $r_1 \dots, r_k$ are unique, being the solution of a system of linear equations: $a_\nu = r_1 \epsilon_1^\nu + \dots + r_k \epsilon_k^\nu (\nu = 0, 1, 2, \dots, k-1)$. Hence $f(z)$ is unique.

5. The original proof of Theorem 2 depends on the following

Theorem 3 (Carathéodory).[8] *Let* $a_1, a_2, \ldots, a_n$ *be any given* n *complex numbers, then* $a_\nu (\nu=1, 2, \ldots, n)$ *can be expressed in the form:*

$$a_\nu = r_1\varepsilon_1^\nu + \ldots + r_k\varepsilon_k^\nu (k \leqq n,\ r_j > 0,\ |\varepsilon_j| = 1,\ j=1, 2, \ldots, k) \quad (22)$$

and such k, r_j, ε_j *are unique.*

This can be proved simply as follows. Since all the roots of the equation $\delta(x, a_1, \ldots, a_n)=0$ are real, let a_0 be its greatest root, then all characteristic numbers of the Hermitian form $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ are non-negative, hence $H_n(x)$ is positive semi-definite, since $\delta(a_0, a_1, \ldots a_n)=0$. Hence by Theorem 2, there exists a unique

$$f(z)=\frac{a_0}{2}+a_1z+\ldots+a_nz^n (\text{mod. } z^{n+1})=\sum_{\nu=1}^{n}\frac{r_\nu}{2}\cdot\frac{1+\varepsilon_\nu z}{1-\varepsilon_\nu z} (r_\nu>0,\ |\varepsilon_\nu|=1,\ k\leqq n),$$

so that $a_\nu = r_1\varepsilon_1^\nu+\ldots+r_k\varepsilon_k^\nu$. We can prove the uniqueness of k, r_j, ε_j as follows. Suppose that a_ν be expressed in the form (22). We put $a_0=r_1+\ldots+r_k$, then $\delta(a_0, a_1, \ldots, a_n)=0$. Let $f(z)=\sum_{\nu=1}^{n}\frac{r_\nu}{2}\cdot\frac{1+\varepsilon_\nu z}{1-\varepsilon_\nu z}$, then

$$f(z)=\frac{a_0}{2}+a_1z+\ \ldots\ +a_nz^n (\text{mod. } z^{n+1}) \text{ and } \Re f(z)\geqq 0 \text{ in } |z|<1, \quad (23)$$

so that $H_n(x)=\sum_0^n a_{\mu-\nu}x_\nu\bar{x}_\mu$ is positive semi-definite, since $\delta(a, a_1 \ldots, a_n)=0$. Hence a_0 is the greatest root of $\delta(x, a_1, \ldots, a_n)=0$, so that a_0 is unique. Since by Theorem 2, such $f(z)$ as (23) is unique, k, r_j, ε_j are unique.

7. Let $f(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nz^n$ be regular in $|z|<1$, whose real part is not necessarily positive. We put

$$m(\rho)=\underset{|z|=\rho}{\text{Min.}}\ \Re f(z),\quad M(\rho)=\underset{|z|=\rho}{\text{Max.}}\ \Re f(z)\ (0<\rho<1). \quad (24)$$

Then by (14), $g^{(\rho)}\leqq 2m(\rho)$. Similarly $2M(\rho)\leqq G^{(\rho)}$. On the other hand, from (8), $g_n^{(\rho)}\geqq 2m(\rho)$, so that $g^{(\rho)}\geqq 2m(\rho)$, hence $g^{(\rho)}=2m(\rho)$. Similarly $G^{(\rho)}=2M(\rho)$. Hence we have the theorem:[9]

Theorem 4. $g^{(\rho)}=2m(\rho)$, $G^{(\rho)}=2M(\rho)$.

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