43. Note on the theory of conformal representation by meromorphic functions II.*)

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§ 4. *Limitations with regard to pole.*

We turn next our attention to the problem relating to the pole z_∞ and its residue A, supposing that there exists at all a pole of the function $f(z)$, schlicht and meromorphic in the basic circle $|z|<1$. We consider now, for that purpose, again as in the part I the corresponding function $g(\zeta)=f(\zeta^{-1})^{-1}$ and make use of a distortion formula

$$(4.1)\qquad |\lg g'(\zeta)| \leqq \lg \frac{|\zeta|^2}{|\zeta|^2-1} \qquad (|\zeta|>1)$$

discovered first by Grunsky[15] and given later otherwise by Golusin[16] which sharpens a classical theorem

$$(4.2)\qquad |g'(\zeta)| \leqq \frac{|\zeta|^2}{|\zeta|^2-1} \qquad (|\zeta|>1)$$

due to Löwner.[17] The logarithmic function in the left-hand side in (4.1) means, of course, such a branch that vanishes at $\zeta=\infty$. The formula (4.1) is, in reality, more profound than all the others used in this paper, save the previously quoted one due to Landau.

We state now the following proposition.

Theorem. *If $f(z)$ is schlicht in $|z|<1$ and possesses there actually a pole z_∞ with residue A, then we have*

$$(4.3)\qquad \left|\lg \frac{-z_\infty^2}{A}\right| \leqq \lg \frac{1}{1-|z_\infty|^2},$$

and hence especially

$$(4.4)\qquad |z_\infty|^2(1-|z_\infty|^2) \leqq |A| \leqq \frac{|z_\infty|^2}{1-|z_\infty|^2}$$

and

$$(4.5)\quad \arg(-z_\infty^2) + \lg(1-|z_\infty|^2) \leqq \arg A \leqq \arg(-z_\infty^2) - \lg(1-|z_\infty|^2),$$

*) I. Proc. **21** (1945), 269.

15) H. Grunsky, Neue Abschätzungen zur konformen Abbildung ein- und mehrfach zusammenhängender Bereiche. Schriften d. math. Sem. u. d. Inst. f. angew. Math. d. Univ. Berlin **1** (1932/3), 95–140.

16) G. M. Golusin, Ergänzung zur Arbeit „Über die Verzerrungssätze der schlichten konformen Abbildungen". Recueil Math. **2** (**44**) (1938), 685–688 (in Russian).

17) K. Löwner, loc. cit. 9). Cf. also E. Frank, Beiträge zur konformen Abbildung. Inaug.-Diss. Frankfurt (1919).

where the many-valued quantities have to be determined by their suitable values. All the limits are attained by some functions belonging to the family.

The general estimation (4.3) in the theorem is an almost immediate consequence of (4.1). In fact, recalling the relations (2.3) and (2.4), we are led at once to it. Next, the equality occurs there, as is shown also by the aid of Grunsky's extremum functions,[18] only when the function $f(z)$ coïncides with one of the schlicht functions, each of which maps $|z|<1$ conformally onto a domain, obtained by cutting the whole plane along an arc of a logarithmic spiral intersecting all the rays, which start from the origin, at a definite angle, $\frac{1}{2}(\varphi+\pi)$ say. Such functions are of the form

$$(4.6) \qquad f(z)=\frac{z_\infty z(1-\bar{z}_\infty z)^\varepsilon}{z_\infty - z},$$

where we have put $\varepsilon=e^{i\varphi}$ for the sake of brevity, and we have furthermore exactly

$$(4.7) \qquad \lg\frac{-z_\infty^2}{A}=\varepsilon\lg\frac{1}{1-|z_\infty|^2}$$

for the function (4.6).

The remaining estimations (4.4) and (4.5) are, of course, the special cases of the general one.[19] We need, in fact, merely to take account only of the real and imaginary parts respectively of the quantity put in the absolute value sign in the left- hand side of (4.3). As for the equalities, they hold good, in the first place, in the left- and right-hand sides of (4.4), by (4.6) and (4.7), only for the functions

$$(4.8) \qquad f(z)=\frac{z_\infty z(1-\bar{z}_\infty z)}{z_\infty - z} \quad \text{and} \quad f(z)=\frac{z_\infty z}{(1-\bar{z}_\infty z)(z_\infty - z)}$$

respectively, which are, of course, derived from (4.6) by putting $\varepsilon=1$ and $\varepsilon=-1$ respectively. The former function maps $|z|<1$ onto the whole plane cut along an arc of the circle of radius $|z_\infty|$ about the origin, the arc having the end-points at

$$(2|z_\infty|^2-1\pm 2i|z_\infty|\sqrt{1-|z_\infty|^2})\,z_\infty=-e^{\mp 2i\arcsin|z_\infty|}\,z_\infty,$$

so that its angle at centre is equal to $4\arcsin|z_\infty|$, where the arcsine-function is supposed as usual to attain its principal values; while the latter, being of the same

18) H. Grnnsky, loc. cit. 15).

19) The left inequality of (4.4) follows, too, from the weaker distortion theorem (4.2). The inequality (4.5) follows correspondingly fr:m a less general form of (4.1) in which arg stands in the place of lg. G. M. Golusin himself showed previously in his paper: Über die Verzerrungssätze der schlichten konf:rmen Abbildungen. Recueil Math. **1** (**43**) (1936), 127-135 (in Russian) this special case only, and later the general one cited in the text, by means of tha so-called Löwner's differential equation which he is fond of using.

form as the already cited function (3.12) or (3.17) and coïnciding, in fact, entirely with it for

$$a_2=\bar{z}_\infty+\frac{1}{z_\infty} \quad \text{and} \quad \varepsilon^2=\frac{a_2^2}{|a_2|^2}=\frac{z_\infty}{\bar{z}_\infty},$$

transforms $|z|<1$ onto the whole plane cut along the segment of length $\dfrac{4|z_\infty|}{(1-|z_\infty|^2)^2}$ between the two points $\dfrac{-z_\infty}{(1\pm|z_\infty|)^2}$.

Secondly, all the arguments in (4.5), and the logarithm contained in (4.3) also, relate to such ones as the quantity $\arg A-\arg(-z_\infty^2)$ tends to zero if we suppose that z_∞ itself does so in a continuous manner.[20] In (4.4), the equalities in the left and right sides appear only for the functions

$$f(z)=\frac{z_\infty z(1-\bar{z}_\infty z)^{\pm i}}{z_\infty-z}=\frac{z_\infty z}{z_\infty-z}\exp(\pm i \lg(1-\bar{z}_\infty z))$$

respectively, which follow again from (4.6) by putting $\varepsilon=\pm i$ and where the logarithmic function $\lg(1-\bar{z}_\infty z)$ in the exponent are assumed to denote its branch vanishing at $z=0$. Each of these extremum functions maps $|z|<1$ onto the whole plane cut along an arc of a logarithmic spiral intersecting all the rays, which start from the origin, at the angle $\dfrac{\pi}{2}\pm\dfrac{\pi}{4}$ respectively. The end-points of the arcs in question are laid at the images of the original points

$$\frac{1-i}{2}\frac{z_\infty}{|z_\infty|}(-|z_\infty|\pm i\sqrt{2-|z_\infty|^2}) \quad \text{and} \quad \frac{1+i}{2}\frac{z_\infty}{|z_\infty|}(-|z_\infty|\mp i\sqrt{2-|z_\infty|^2})$$

respectively, all of which lie on $|z|=1$ and are, in reality, the zero points of the derivatives of the respective mapping functions.

On this occasion, it may be noticed that the estimations (4.4) for $|A|$ by means of $|z_\infty|$ can conversely be regarded as those for $|z_\infty|$ by means of $|A|$ In fact, solving these inequalities with regard to $|z_\infty|$, we have first

$$|z_\infty|\geqq\sqrt{\frac{|A|}{1+|A|}},$$

at any time, and on the other hand

$$|z_\infty|\leqq\frac{1}{2}(\sqrt{1+2\sqrt{|A|}}-\sqrt{1-2\sqrt{|A|}})$$

or

$$|z_\infty|\geqq\frac{1}{2}(\sqrt{1+2\sqrt{|A|}}+\sqrt{1+2\sqrt{|A|}})$$

provided that $4|A|\leqq1$. All the limits are attained by the respective functions.

§ 5. *Some coefficient problems.*

Let the function $f(z)$ be, here also, meromorphic and schlicht in $|z|<1$ and normalized at $z=0$. Suppose further that there exists, as before, actually a

20) Cf. also the relatious (5.7) stated later.

pole z_∞ with residue A. Then, supposing the function $f(z)-\dfrac{A}{z-z_\infty}$ regular in $|z|<1$ to be expanded in power series about $z=0$, we put

$$(5.1) \qquad f(z)=\frac{A}{z-z_\infty}+\sum_{n=0}^{\infty}\alpha_n z^n \qquad (|z|<1).$$

The original expansion of the function itself being

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,$$

say, with normalization conditions

$$a_0=f(0)=0 \quad \text{and} \quad a_1=f'(0)=1,$$

we have immediately

$$(5.2) \qquad \alpha_0=\frac{A}{z_\infty}, \quad \alpha_1=\frac{A}{z_\infty^2}+1; \quad \alpha_n=\frac{A_\infty}{z_\infty^{n+1}}+a_n \ (n>1).$$

We can then, in conformity with such circumstances, deduce some. conditions to which the coefficients in (5.2) must be subject. We have namely, as an immediate consequence of the preceding results, the following proposition.

Theorem. *Let $f(z)$ denote a function of the same kind as in the preceding theorem and further (5.1) its expression. Then we have, with respect to the beginning coefficients,*

$$(5.3) \qquad \left|\lg\frac{-z_\infty^2}{\alpha_0}\right|=\left|\lg\frac{1}{1-\alpha_1}\right|\leqq \lg\frac{1}{1-|z_\infty|^2}$$

and

$$(5.4) \qquad \left|\alpha_2-\frac{A}{z_\infty^3}\right|\leqq |z_\infty|+\frac{1}{|z_\infty|},$$

all these estimations being precise.

In fact, the inequality (5.3) follows at once, by remembering the relations (5.2), from the preceding theorem, i.e. (4.3), and the inequality (5.4) also analogously by recalling (3.9). To show that the former gives a precise limitation, we have only to consider the extremum functions (4.6) with coefficients

$$\alpha_0=-z_\infty^2 e^{\varepsilon\lg(1-|z_\infty|^2)} \quad \text{and} \quad \alpha_1=1-e^{\varepsilon\lg(1-|z_\infty|^2)},$$

for and only for which the equality in (5.3) actually occurs. Next, in (5.4), the second function in (4.8) which represents the special case $\varepsilon=-1$ of those, stands for the unique extremum function and we have indeed for it

$$A=-\frac{z_\infty^2}{1-|z_\infty|^2}, \qquad \alpha_n=-\frac{|z_\infty|^2\bar{z}_\infty^{n-1}}{1-|z_\infty|^2} \ (n>1);$$

$$\alpha_2-\frac{A}{z_\infty^3}=\bar{z}_\infty+\frac{1}{z_\infty}=\frac{\bar{z}_\infty}{|z_\infty|}\left(|z_\infty|+\frac{1}{|z_\infty|}\right).$$

We can also, for the remaining coefficients of the series in (5.2), though rough enough, derive inequalities of analogous form. For example, as the function

$$\frac{f(z_\infty z)}{z_\infty}=z+\sum_{n=2}^{\infty} z_\infty^{n-1}a_n z^n$$

is at any rate regular and schlicht in $|z|<1$, we have generally, by a Littlewood's coefficient theorem,[21] for any $n>1$,

$$(5.5)\quad \left|a_n-\frac{A}{z_\infty^{n+1}}\right|=|a_n|<\left(1+\frac{1}{n-1}\right)^{n-1}\frac{n}{|z_\infty|^{n-1}}<\frac{en}{|z_\infty|^{n-1}}.$$

Now, we can find, if desired, limitations for $|a_n|$ depending only on $|z_\infty|$. We have, for instance, from (5.3), (5.4) and (5.5), by taking (4.4) into account,

$$(5.6)\quad \begin{cases} |a_0|\leqq\dfrac{|z_\infty|}{1-|z_\infty|^2}, \quad |a_1|\leqq\dfrac{2-|z_\infty|^2}{1-|z_\infty|^2}, \quad |a_2|\leqq\dfrac{1}{|z_\infty|}\dfrac{2-|z_\infty|^4}{1-|z_\infty|^2}, \\ |a_n|<\dfrac{1}{|z_\infty|^{n-1}}\left(en+\dfrac{1}{1-|z_\infty|^2}\right) \quad (n>2) \end{cases}$$

respectively. But all the limits in (5.5) and (5.6), save only one for a_0, not precise at all.

We note on this occasion, by means of (4.3) and (5.3), that the interesting *asymptotic behaviours*

$$(5.7)\quad A\equiv -z_\infty^2+O(|z_\infty|^4),\quad a_0\equiv -z_\infty+O(|z_\infty|^3),\quad a_1=O(|z_\infty|^2)$$

hold always uniformly provided that the pole z_∞ lies near the zero point 0 of the function.

We consider next, in place of (5.1), the Laurent expansion of the function itself about its pole z_∞,

$$(5.8)\qquad f(z)\equiv\frac{A}{z-z_\infty}+\sum_{n=0}^{\infty}\beta_n(z-z_\infty)^n$$

say, which is certainly valid e.g. in the annular domain $0<|z-z_\infty|<1-|z_\infty|$. In this case we are also able to deduce the precise limits concerning the beginning coefficients β_0 and β_1.

Theorem. *We have always, and precisely, the necessary conditions*

$$(5.9)\qquad \left|\frac{1-|z_\infty|^2}{A}\beta_0+\bar{z}_\infty\right|\leqq\frac{1}{|z_\infty|}+|z_\infty|$$

and

$$(5.10)\qquad \left|\frac{\beta_1}{A}\right|\leqq\frac{1}{(1-|z_\infty|^2)^2}.$$

To prove the proposition, we introduce here an intermediate variable ζ by means of the linear transformation

$$\zeta=\frac{1-\bar{z}_\infty z}{z_\infty-z}\qquad\text{or}\qquad z=\frac{z_\infty\zeta-1}{\zeta-\bar{z}_\infty},$$

21) J. E. Littlewood, On inequalities in the theory of functions. Proc. London Math. Soc. **23** (1925), 481-519.

which transforms $|z|<1$ onto $|\zeta|>1$ in such a way that the two pointe $z=z_\infty$ and $z=0$ correspond to $\zeta=\infty$ and $\zeta=\dfrac{1}{z_\infty}$ respectively, and then define a new function $H(\zeta)$ by the expression

$$(5.11)\qquad H(\zeta)=-\frac{1-|z_\infty|^2}{A}f\left(\frac{z_\infty\zeta-1}{\zeta-\bar{z}_\infty}\right).$$

This function is certainly schlicht on $|\zeta|>1$ and normalized at $\zeta=\infty$, i.e. of the same form, in a neighbourhood of $\zeta=\infty$, as the function $g(\zeta)$ given in (3.1) and, moreover, its Laurent expansion about this point begins with

$$(5.12)\qquad H(\zeta)=\zeta-\left(\frac{1-|z_\infty|^2}{A}\beta_0+z_\infty\right)+\frac{(1-|z_\infty|^2)^2}{A}\frac{\beta_1}{\zeta}+\dots.$$

Hence, the Löwner's distortion formula (3.3), applied to the function $H(\zeta)$ with respect to the point $\zeta=\dfrac{1}{z_\infty}$, leads us, since $H\left(\dfrac{1}{z_\infty}\right)=0$, immediately to (5.9), while the area theorem (3.10) implies particularly (5.10). The second function in (4.8) stands evidently again for the unique extremum function in both inequalities (5.9) and (5.10), and possesses actually the attributes

$$A=-\frac{z_\infty^2}{1-|z_\infty|^2},\qquad \beta_n=-\frac{|z_\infty|^2\bar{z}_\infty^{\,n-1}}{(1-|z_\infty|^2)^{n+2}}\quad(n\geqq 0),$$

$$H(\zeta)=\zeta-\left(\bar{z}_\infty+\frac{1}{z_\infty}\right)+\frac{\bar{z}_\infty}{z_\infty}\frac{1}{\zeta};$$

this completes the proof.

If we require again some estimations, with bounds depending only upon the quantity $|z_\infty|$, for the coefficients themselves in question, we may take, for instance, the right inequality in (4.4) into account; namely, it follows then, by combining it with (5.9) and (5.10), that

$$(5.12)\qquad |\beta_0|\leqq\frac{|z_\infty|(1+2|z_\infty|^2)}{(1-|z_\infty|^2)^2}$$

and

$$(5.13)\qquad |\beta_1|\leqq\frac{|z_\infty|^2}{(1-|z_\infty|^2)^3}$$

respectively. Though the equality is never realized in (5.12), the limit in (5.13) is surely attained again by the just mentioned extremum function.

§ 6, *Further discussions on limitations concerning pole*.

In conclusion, we now notice, in order to fulfil the promise stated at the footnote[13] of § 3, that various inequalities obtained in the last paragraph are of the form analogous to (3.9) from which the result (3.8) has followed. Similarly, we can hence regard these inequalities as those which give conversely the limitations concerning the position of the pole z_∞ or its residue A, provided that the respective coefficients are preassigned. We illustrate these circumstances by the following theorem taken as an example.

Theorem. *Retaining all the notations used above, we conclude that, given previously the magnitude of the coefficiento* a_0, *the position of the pole* z_∞ *is restricted by*

$$|z_\infty| \geqq \frac{2|a_0|}{\sqrt{1+4|a_0|^2}+1} \tag{6.1}$$

and

$$|z_\infty| \geqq \sqrt{1-\mathrm{Min}\left(|1-a_1|, \frac{1}{|1-a_1|}\right)} \equiv \sqrt{1-\exp(-|\lg|1-a_1||)}, \tag{6.2}$$

and, for given $|\beta_1|$, *by*

$$|z_\infty| \geqq m(|\beta_1|), \tag{6.3}$$

where $m=m(\beta)$ $(\beta \geqq 0)$ *denotes the uniquely determined root of the equation*

$$m^2=\beta(1-m^2)^3, \qquad 0 \leqq m < 1,$$

and on the other hand that the residue by

$$|A| \geqq \frac{2|a_0|^2}{\sqrt{1+4|a_0|^2}+1} = \frac{1}{2}(\sqrt{1+4|a_0|^2}-1). \tag{6.4}$$

In fact, all these results are immediate consequences of (5.3) and (5.13) and further of (5.2) respectively. It is worthy to note that all these inequalities (6.1), (6.2), (6.3) and (6.4) possess a common extremum function, viz. the second function in (4.8).