30. *Conics in D van Dantzig's projective space.*

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§ 0. In his very interesting papers, H. Hombu[1] has developed the projective theory of a system of paths of higher order. He has also treated, as an application of his general theory, the system of paths of the third order defined by the differential equations of the form

$$T^i \equiv x^{(3)i} + A^i_k x^{(2)k} + B^i = 0^{2)}, \qquad (i, j, k, \ldots = 1, 2, \ldots, n) \tag{0.1}$$

where $A^i_k(x, x^{(1)})$ and $B^i(x, x^{(1)})$ are homogeneous functions of the degree $+1$ and $+3$ respectively in $x^{(1)i}$, $x^{(r)i}$ denoting the ordinary r-th derivative with respect to the parameter chosen.

On the other hand, M. Mikami[3] has studied the parabolas in the so-called generalized spaces, say, in the spaces of line-elements $(x, x^{(1)})$ of the first order with an affine connection $\Gamma^i_{jk}(x, x^{(1)})$, the parallel displacement of a vector v^i being defined by the vanishing of the covariant differential

$$\delta v^i = dv^i + \Gamma^i_{jk} v^j dx^k. \tag{0.2}$$

M. Mikami has defined parabolas by the differential equations of the form

$$\frac{\delta^2}{ds^2} x^{(1)i} = 0, \tag{0.3}$$

as a natural generalization of parabolas in an ordinary affine space. If we write down fully the equations (0.3), we obtain the equations of the form (0.1). Then, what is the necessary and sufficient condition that the system of paths (0.1) defines a system of parabolas? The answer to this question was also given by M. Mikami.

H. Hombu and M. Mikami[4] have continued this study of parabolas in the generalized spaces of paths of J. Douglas. They have considered the contacts of

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1) H. Hombu: Die projektive Theorie eines Systems der "paths" höherer Ordnung, I, Japanese Journal of Math., **15** (1938), 139–196; II, Journ. Fac. Sc. Hokkaido Imp. Univ., (I) **7** (1938), 35–94.

2) H. Hombu: Die projektive Theorie der "paths" $x^{(3)i} + A^i_k x^{(2)i} + B^i = 0$. Proc., **13** (1937), 410–413.

3) M. Mikami: On parabolas in the generalized spaces. Japanese Journal of Math., **17** (1940), 185–200.

4) H. Hombu and M. Mikami: Parabolas and projective transformations in the generalized spaces of paths. Japanese Journal of Math., **17** (1941), 307–335.

parabolas in two projectively related spaces and common parabolas in two such spaces and determined projectively related spaces having common parabolas.

They have also studied the conics[1] in the projectively connected space of T. Y. Thomas: they have defined the conics by the differential equations of the form

$$(0.3)\qquad \frac{\delta^2}{dt^2}x^{(1)\lambda}=0\,, \qquad (\lambda, \mu, \nu, \dots =0, 1, 2, \dots, n)$$

and studied the projective parameters on conics, the contacts of two conics and the contacts of the parabolas and conics.

On the other hand, the present authors[2] have studied the conics in affinely or projectively connected spaces. It can be summarized as follows:

1°. We consider first of all the affine conics. Consider an affinely connected space, referred to a coodinate system (x^i) and with the connection parameters $\Gamma^i_{jk}(x)$. Then, if we are given a curve $x^i(r)$ in this space, we can develop, in an ordinary affine space, this curve and the reference frames attached to the every point of the curve.

If we obtain, after the development, a plane curve in the ordinary affine space, we call the original curve plane curve in the affinely connected space. The defining equations of the plane curves are

$$(0.5)\qquad \frac{\delta^3 x^i}{dr^3}+\alpha\frac{\delta^2 x^i}{dr^2}+\beta\frac{dx^i}{dr}=0\,,$$

where δ/dr denotes the covariant differentiation along the curve with respect to the affine connection Γ^i_{jk}, α and β being certain functions of the parameter r.

If we choose a suitable parameter s on the curve, the differential equations (0.5) of the curve may be reduced to the more simple form

$$(0.6)\qquad \frac{\delta^3 x^i}{ds^3}+k\frac{dx^i}{ds}=0\,,$$

where k is a function of s. The parameters which give the simple form (0.6) to the differential equations of a plane curve being related to each other by an equation of the form $\bar{s}=as+b$, a and b being constants, we may call it affine parameters on the plane curve. We define the affine normal of a plane curve as the direction given by $\dfrac{\delta^2 x^i}{ds^2}$.

If we suppose that the affine normals of a plane curve are concurrent, we must have

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1) H. Hombu and M. Mikami: Conics in the projectively connected manifolds. Memoires of the Fac. Sc., Kyûsyû Imp. Univ., **2** (1941), 217–239.

2) K. Yano and K. Takano: Sur les coniques dans les espaces à connexion affine ou projective, I. Proc., **20** (1944), 410–417; II. ibidem, 418–424.

$$\frac{dx^i}{ds}+\frac{\delta}{ds}\left(\lambda\frac{\delta^2x^i}{ds^2}\right)=0\,, \tag{0.7}$$

λ being a certain function of s. Thus, from the equations (0.6) and (0.7), we know that the function λ and consequently k must be constant.

We call such a curve conic in the affinely connected space. If $k=0$, our conic becomes a parabola defined by M. Mikami.

2°. To consider the conics in projectively connected space, we take Cartan's repère semi-natural $[A_0, A_1, \cdots A_n]$, the projective connection being defined by

$$dA_0=dx^0A_0+dx^iA_i\,,\quad dA_j=\Gamma^0_{jk}dx^kA_0+\Gamma^i_{jk}dx^kA_i\,, \tag{0.8}$$

where x^0 is a non holonomic variable.

Then, the conics are defined by the eqnation of the form

$$\frac{d^3}{dt^3}\rho A_0=0\,, \tag{0.9}$$

as a natural generalization of ordinary projective conics, where ρ is a suitable function of the parameter t.

Introducing an affine parameter s, we have, from (0.9), by a straightforward calculation,

$$\begin{cases}\dfrac{da^0}{ds}+\Gamma^0_{jk}a^j\dfrac{dx^k}{ds}+\dfrac{d}{ds}\{t,s\}=0\,,\\[2ex] \dfrac{da^i}{ds}+\Gamma^i_{jk}a^j\dfrac{dx^k}{ds}+(a^0+2\{t,s\})\dfrac{dx^i}{ds}=0\,,\end{cases} \tag{0.10}$$

where

$$a^0=\Gamma^0_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\,,\quad a^i=\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\,,$$

and $\{t, s\}$ denotes the Schwarzian derivative of t with respect to the affine parameter s. These equations show that t is a projective parameter on the conic. The non holonomic variable x^0 is defined by

$$x^0=\log\left(\frac{1}{\rho}\frac{dt}{ds}\right) \tag{0.11}$$

§ 1. J. Haantjes[1] has discussed a few years ago the projective geometry of paths with the aide of D. van Dantzig's homogeneous curvilinear coordinates (x^λ). The components of the projective connection $\Pi^\lambda_{\mu\nu}$ satisfying the three conditions

$$\Pi^\lambda_{\mu\nu}=\Pi^\lambda_{\nu\mu}\,,\quad \Pi^\lambda_{\mu\nu}x^\mu=0\,,\quad \Pi^\lambda_{\mu\nu}(\rho x)=\rho^{-1}\Pi^\lambda_{\mu\nu}(x), \tag{1.1}$$

the paths in D. van Dantzig's projective space are defined by the differential equations

1) J. Haantjes: On the projective geometry of paths. Proc. Edinburgh Math. Soc., 5 (1937), 103–115.

(1.2) $$\frac{d^2x^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha\frac{dx^\lambda}{dr}+\beta x^\lambda,$$

α and β being certain functions of the parameter r.

On the other hand, one of the present author K. Yano[1] has shown that we can write, in this space, the differential equations of the paths in the form

(1.3) $$\frac{\delta^2}{dt^2}(\rho x^\lambda)=0\,,$$

choosing a suitable parameter t and its function ρ. The parameter t is a projective one.

To compare J. Haantjes' theory with L. Berwald's one,[2] and with K. Yano's one,[3] we introduce a system of non homogeneous coordinates ξ^i by the equations

(1.4) $$\xi^i=\xi^i(x^\lambda)\,,$$

$\xi^i(x)$ being homogeneous functions of degree zero, so that satisfying

(1.5) $$E^i_{.\lambda}x^\lambda=0\,,\qquad \left(E^i_{.\lambda}=\frac{\partial\xi^i}{\partial x^\lambda}\right)$$

where the rank of the matrix $(E^i_{.\lambda})$ is supposed to be n.

In order to define the inverse of the matrix $(E^i_{.\lambda})$, we introduce a covariant projective vector p_λ satisfying

(1.6) $$p_\lambda x^\lambda=1\,,$$

the components p_λ being homogeneous functions of x^λ of degree -1.

Theh, the inverse $(E^{.\lambda}_j)$ of $(E^i_{.\lambda})$ is defined by means of the relations

(1.7) $$E^{.\lambda}_j E^i_{.\lambda}=\delta^i_j \qquad\text{and}\qquad E^{.\lambda}_j p_\lambda=0\,.$$

Now, we define the quantities Γ^0_{jk} and Γ^i_{jk} by means of

(1.8) $$\Gamma^0_{jk}=-E^{.\mu}_j E^{.\nu}_k(p_{\mu,\nu}-p_\lambda\Pi^\lambda_{\mu\nu})$$

and

(1.9) $$\Gamma^i_{jk}=-E^{.\mu}_j E^{.\nu}_k(E^i_{.\mu,\nu}-E^i_{.\lambda}\Pi^\lambda_{\mu\nu})$$

respectively, the comma denoting the ordinary partial derivative with respect to the homogeneous coordinates x^λ.

Then, the equation of paths (1.3) gives us

(1.10) $$\{t,s\}=-2\Gamma^0_{jk}\frac{d\xi^j}{ds}\frac{d\xi^k}{ds}$$

and

1) K. Yano: Les espaces à connexion projective et la géométrie projective des paths. Annales Scientifiques de l'Université de Jassy, **24** (1938), 295–464.

2) L. Berwald: On the projective geometry of paths. Annals of Math., **37** (1936), 879–898.

3) K. Yano: Projective parameters on paths in D. van Dantzig's projective space. Proc., **20** (1944), 210–215.

$$\frac{d^2\xi^i}{ds^2}+\Gamma^i_{jk}\frac{d\xi^j}{ds}\frac{d\xi^k}{ds}=0\,. \tag{1.11}$$

These equations show that Γ^0_{jk} are components of a tensor defining the projective parameters on paths and Γ^i_{jk} are those of the affine connection which give the same system of paths as the projective connection $\Pi^\lambda_{\mu\nu}$.

Thus, the functions $(\Gamma^0_{jk}, \Gamma^i_{jk})$, may be considered as defining a projective connection.

Now, it will be easily proved that the following relations hold good:

$$E_i^{\cdot\lambda}E^i_{\cdot\nu}=\delta^\lambda_\nu-p_\nu x^\lambda\,, \tag{1.12}$$

$$p_{\mu,\nu}-p_\lambda\Pi^\lambda_{\mu\nu}+p_\mu p_\nu=-E^j_{\cdot\mu}E^k_{\cdot\nu}\Gamma^0_{jk}\,, \tag{1.13}$$

$$E^i_{\cdot\mu,\nu}-E^i_{\cdot\lambda}\Pi^\lambda_{\mu\nu}+p_\mu E^i_{\cdot\nu}+p_\nu E^i_{\cdot\mu}=-E^j_{\cdot\mu}E^k_{\cdot\nu}\Gamma^i_{jk}\,. \tag{1.14}$$

§ 2. The purpose of the present Note is to study the conics in D. van Dantzig's projective space using the results of the previous section.

In D. van Dantzig's projective space, we take the homogeneous curvilinear coordinates x^λ, and consider a curve $x^\lambda(t)$ satisfying

$$\frac{\delta^3}{dt^3}\rho x^\lambda=0\,, \tag{2.1}$$

where t is a projective parameter and ρ is its suitable function, δ/dt denoting the covariant differentiation with respect to the projective connection $\Pi^\lambda_{\mu\nu}$ satisfying (1.1).

Let us call such a curve $x^\lambda(t)$ satisfying (2.1) the conic in D. van Dantzig's projective space. In the following lines, we shall consider the equations of conics thus defined in the non homogeneous coordinate system (ξ^i) using the components of projective connection $(\Gamma^0_{jk}, \Gamma^i_{jk})$.

Differentiating $\xi^i(t)=\xi^i(x^\lambda(t))$ with respect to the parameter t, we have successively

$$\frac{d\xi^i}{dt}=E^i_{\cdot\lambda}\frac{dx^\lambda}{dt}\,, \tag{2.2}$$

$$\frac{d^2\xi^i}{dt^2}=E^i_{\cdot\mu,\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}+E^i_{\cdot\lambda}\frac{d^2x^\lambda}{dt^2}\,. \tag{2.3}$$

Substituting in (2.3) $E^i_{\cdot\mu,\nu}$ obtained from (1.14), and taking account of (2.2), we find

$$\frac{d^2\xi^i}{dt^2}+\Gamma^i_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt}=E^i_{\cdot\lambda}\Big(\frac{d^2x^\lambda}{dt^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\Big)-2p_\lambda\frac{dx^\lambda}{dt}\frac{d\xi^i}{dt}\,,$$

or

$$\frac{\delta^2\xi^i}{dt^2}=E^i_{\cdot\lambda}\frac{\delta^2x^\lambda}{dt^2}-2p_\lambda\frac{dx^\lambda}{dt}\frac{d\xi^i}{dt}\,, \tag{2.4}$$

putting

$$\frac{\delta^2\xi^i}{dt^2}=\frac{d^2\xi^i}{dt^2}+\Gamma^i_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt} \quad \text{and} \quad \frac{\delta^2x^\lambda}{dt^2}=\frac{d^2x^\lambda}{dt^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}.$$

Differentiating (2.4) once more with respect to t, we have

$$\frac{d}{dt}\frac{\delta^2\xi^i}{dt^2}=E^i_{\cdot\mu,\nu}\frac{\delta^2x^\mu}{dt^2}\frac{dx^\nu}{dt}+E^i_{\cdot\lambda}\frac{d}{dt}\frac{\delta^2x^\lambda}{dt^2}$$

$$-2p_{\mu;\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{d\xi^i}{dt}-2p_\lambda\frac{\delta^2x^\lambda}{dt^2}\frac{d\xi^i}{dt}-2p_\lambda\frac{dx^\lambda}{dt}\frac{d^2\xi^i}{dt^2},$$

where

$$p_{\mu;\nu}=p_{\mu,\nu}-p_\lambda\Pi^\lambda_{\mu\nu}.$$

Substituting in the above equation $p_{\mu;\nu}$ and $E^i_{\cdot\mu,\nu}$ obtained from (1.13) and (1.14) respectively, we find

$$\frac{d}{dt}\frac{\delta^2\xi^i}{dt^2}=E^i_{\cdot\lambda}\frac{\delta^3x^\lambda}{dt^3}-3p_\lambda\frac{\delta^2x^\lambda}{dt^2}\frac{d\xi^i}{dt}-p_\lambda\frac{dx^\lambda}{dt}E^i_{\cdot\mu}\frac{\delta^2x^\mu}{dt^2}-E^j_{\cdot\mu}\frac{\delta^2x^\mu}{dt^2}\Gamma^i_{jk}\frac{d\xi^k}{dt}$$

$$+2\Gamma^0_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt}\frac{d\xi^i}{dt}+2\left(p_\lambda\frac{dx^\lambda}{dt}\right)^2\frac{d\xi^i}{dt}-2p_\lambda\frac{dx^\lambda}{dt}\frac{d^2\xi^i}{dt^2}.$$

Substituting $E^i_{\cdot\lambda}\dfrac{\delta^2x^\lambda}{dt^2}$ obtained from (2.4) in these equations, we find

$$\frac{\delta^3\xi^i}{dt^3}=E^i_{\cdot\lambda}\frac{\delta^3x^\lambda}{dt^3}-3p_\lambda\frac{\delta^2x^\lambda}{dt^2}\frac{d\xi^i}{dt}-3p_\lambda\frac{dx^\lambda}{dt}\frac{\delta^2\xi^i}{dt^2}$$
$$+2\Gamma^0_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt}\frac{d\xi^i}{dt}. \tag{2.5}$$

Now, from (2.1), we have

$$\frac{\delta^3x^\lambda}{dt^3}=-3\frac{\rho'}{\rho}\frac{\delta^2x^\lambda}{dt^2}-3\frac{\rho''}{\rho}\frac{dx^\lambda}{dt}-\frac{\rho'''}{\rho}x^\lambda_{\cdot}, \tag{2.6}$$

where the dash represents the ordinary differentiation with respect to the projective parameter t. Substituting (2.6) in (2.5), we find

$$\frac{\delta^3\xi^i}{dt^3}=-3\frac{\rho'}{\rho}E^i_{\cdot\lambda}\frac{\delta^2x^\lambda}{dt^2}-3\frac{\rho''}{\rho}\frac{d\xi^i}{dt}-3p_\lambda\frac{\delta^2x^\lambda}{dt^2}\frac{d\xi^i}{dt}-3p_\lambda\frac{dx^\lambda}{dt}\frac{\delta^2\xi^i}{dt^2}$$
$$+2\Gamma^0_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt}\frac{d\xi^i}{dt},$$

or, taking account of (2.4),

$$\frac{\delta^3\xi^i}{dt^3}+3\left(\frac{\rho'}{\rho}+p_\lambda\frac{dx^\lambda}{dt}\right)\frac{\delta^2\xi^i}{dt^2}$$
$$+\left(3\frac{\rho''}{\rho}+6\frac{\rho'}{\rho}p_\lambda\frac{dx^\lambda}{dt}+3p_\lambda\frac{\delta^2x^\lambda}{dt^2}\right.$$
$$\left.-2\Gamma^0_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt}\right)\frac{d\xi^i}{dt}=0. \tag{2.7}$$

Putting

$$(2.8)\qquad \alpha=3\left(\frac{\rho'}{\rho}+p_\lambda\frac{dx^\lambda}{dt}\right),$$

$$(2.9)\qquad \beta=3\frac{\rho''}{\rho}+6\frac{\rho'}{\rho}p_\lambda\frac{dx^\lambda}{dt}+3p_\lambda\frac{\delta^2x^\lambda}{dt^2}-2\Gamma^0_{jk}\frac{d\xi^j}{dt}\frac{d\xi^k}{dt},$$

we have, from (2.7),

$$(2.10)\qquad \frac{\delta^3\xi^i}{dt^3}+\alpha\frac{\delta^2\xi^i}{dt^2}+\beta\frac{d\xi^i}{dt}=0.$$

Now, we introduce here a parameter s arbitary for a moment and change the parameter t into s and denote by a dot the ordinary differentiation with respect to s. Then, we have successively

$$\frac{d\xi^i}{dt}=\frac{1}{\dot t}\frac{d\xi^i}{ds},$$

$$\frac{\delta^2\xi^i}{dt^2}=\frac{1}{\dot t^2}\frac{\delta^2\xi^i}{ds^2}-\frac{\ddot t}{\dot t^3}\frac{d\xi^i}{ds},$$

$$\frac{\delta^3\xi^i}{dt^3}=\frac{1}{\dot t^3}\frac{\delta^3\xi^i}{ds^3}-3\frac{\ddot t}{\dot t^4}\frac{\delta^2\xi^i}{ds^2}-\left(\frac{\dddot t}{\dot t^4}-3\frac{\ddot t^2}{\dot t^5}\right)\frac{d\xi^i}{ds}.$$

Substituting these equations in (2.10), we find

$$(2.11)\qquad \frac{\delta^3\xi^i}{ds^3}+\left(\alpha\dot t-3\frac{\ddot t}{\dot t}\right)\frac{\delta^2\xi^i}{ds^2}+\left(3\frac{\ddot t^2}{\dot t^2}-\frac{\dddot t}{\dot t}-\alpha\ddot t+\beta\dot t^2\right)\frac{d\xi^i}{ds}=0.$$

We shall here determine the parameter s by the condition

$$(2.12)\qquad \alpha=3\ddot t/\dot t^2,$$

then, the equations (2.11) become

$$(2.13)\qquad \frac{\delta^3\xi^i}{ds^3}+\left(\beta\dot t^2-\frac{\dddot t}{\dot t}\right)\frac{d\xi^i}{ds}=0.$$

The parameter introduced here may be called an affine parameter, for, if a transformation of the parameter s into $\bar s$ keeps the absence of the term $\frac{\delta^2\xi^i}{ds^2}$ in (2.13), the parameters s and $\bar s$ are related by the equation of the form $\bar s=as+b$.

For the purpose of giving one-to-one correspondence between the systems of coordinates (x^λ) and (ξ^i), we have only to introduce an extra variable ξ^0 by the equation

$$(2.14)\qquad d\xi^0=p_\lambda dx^\lambda,$$

the variable ξ^0 being a non holonomic one.

Then, disregarding the integral constants, we have, from (2.8) and (2.12),

$$(2.15)\qquad \xi^0=\log\left(\frac{1}{\rho}\frac{dt}{ds}\right).$$

Now, we shall calculate the coefficient of $\frac{d\xi^i}{ds}$ in (2.13). For this purpose, we write down first the β with the use of the parameter s:

$$(2.16)\qquad \beta=3\frac{\ddot{\rho}}{\rho\dot{t}^2}-3\frac{\dot{\rho}\ddot{t}}{\rho\dot{t}^3}+6\frac{\dot{\rho}}{\rho\dot{t}^2}p_\lambda\frac{dx^\lambda}{ds}+\frac{3}{\dot{t}^2}p_\lambda\frac{\delta^2x^\lambda}{ds^2}$$
$$-3\frac{\ddot{t}}{\dot{t}^3}p_\lambda\frac{dx^\lambda}{ds}-\frac{2}{\dot{t}^2}\Gamma^0_{jk}\frac{d\xi^j}{ds}\frac{d\xi^k}{ds}.$$

On the other hand, we have, from the equations (2.8) and (2.12),

$$(2.17)\qquad p_\lambda\frac{dx^\lambda}{ds}=\frac{\ddot{t}}{\dot{t}}-\frac{\dot{\rho}}{\rho},$$

from which we have by differentiation

$$p_{\mu;\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}+p_\lambda\frac{\delta^2x^\lambda}{ds^2}=\frac{\dddot{t}}{\dot{t}}-\frac{\ddot{t}^2}{\dot{t}^2}-\frac{\ddot{\rho}}{\rho}+\frac{\dot{\rho}^2}{\rho^2}.$$

Substituting in this equation $p_{\mu;\nu}$ obtained from (1.13), we have

$$-\Gamma^0_{jk}\frac{d\xi^j}{ds}\frac{d\xi^k}{ds}-\left(p_\lambda\frac{dx^\lambda}{ds^2}\right)^2+p_\lambda\frac{\delta^2x^\lambda}{ds^2}=\frac{\dddot{t}}{\dot{t}}-\frac{\ddot{t}^2}{\dot{t}^2}-\frac{\ddot{\rho}}{\rho}+\frac{\dot{\rho}^2}{\rho^2},$$

or

$$(2.18)\qquad p_\lambda\frac{\delta^2x^\lambda}{ds^2}=\Gamma^0_{jk}\frac{d\xi^j}{ds}\frac{d\xi^k}{ds}+\frac{\dddot{t}}{\dot{t}}-2\frac{\dot{\rho}\ddot{t}}{\rho\dot{t}}-\frac{\ddot{\rho}}{\rho}+2\frac{\dot{\rho}^2}{\rho^2}$$

on account of (2.17). Substituting (2.17) and (2.18) in (2.16), we have

$$\beta=\frac{a^0}{\dot{t}^2}+3\frac{\dddot{t}}{\dot{t}^3}-3\frac{\ddot{t}^2}{\dot{t}^4},$$

consequently

$$\beta\dot{t}^2-\frac{\dddot{t}}{\dot{t}}=a^0+2\frac{\dddot{t}}{\dot{t}}-3\frac{\ddot{t}^2}{\dot{t}^2},$$

where

$$(2.19)\qquad a^0=\Gamma^0_{jk}\frac{d\xi^j}{ds}\frac{d\xi^k}{ds}.$$

Thus, we have, from (2.13),

$$\frac{\delta^3\xi^i}{ds^3}+(a^0+2\{t,s\})\frac{d\xi^i}{ds}=0,$$

this is the second equation of (0.10).

§ 3. From the equation

$$(3.1)\qquad p_\lambda\frac{dx^\lambda}{dt}=\frac{t}{\dot{t}^2}-\frac{\dot{\rho}}{\rho\dot{t}},$$

we obtain by differentiation

$$p_{\mu;\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}+p_\lambda\frac{\delta^2x^\lambda}{dt^2}=\frac{\dddot t}{\dot t^3}-\frac{2\ddot t^2}{\dot t^4}-\frac{\ddot\rho}{\rho\dot t^2}+\frac{\dot\rho^2}{\rho^2\dot t^2}+\frac{\rho\ddot t}{\rho\dot t^3}.$$

Substituting (1.13) and (2.17) in this equation, and taking account of (2.17), we have

$$(3.2)\qquad p_\lambda\frac{\delta^2x^\lambda}{dt^2}=\frac{a^0}{\dot t^2}+\frac{\dddot t}{\dot t^3}-\frac{\ddot t^2}{\dot t^4}-\frac{\ddot\rho}{\rho\dot t^2}+2\frac{\dot\rho^2}{\rho^2\dot t^2}-\frac{\dot\rho\ddot t}{\rho\dot t^3}.$$

Differentiating (3.2) once more with respect to t, and taking account of (1.13), we find

$$(-E^{j}_{.\mu}E^{k}_{.\nu}\Gamma^0_{jk}-p_\mu p_\nu)\frac{\delta^2x^\mu}{dt^2}\frac{dx^\nu}{dt}+p_\lambda\frac{\delta^3x^\lambda}{dt^3}$$

$$=\frac{\dot a^0}{\dot t^3}-\frac{2a^0\ddot t}{\dot t^4}+\frac{\ddddot t}{\dot t^4}-\frac{5\ddot t\dddot t}{\dot t^5}+\frac{4\ddot t^3}{\dot t^6}-\frac{\dddot\rho}{\rho\dot t^3}+\frac{5\dot\rho\ddot\rho}{\rho^2\dot t^3}+\frac{\ddot\rho\ddot t}{\rho\dot t^4}-\frac{4\dot\rho^3}{\rho^3\dot t^3}$$

$$-\frac{3\dot\rho^2\ddot t}{\rho^2\dot t^4}-\frac{\dot\rho\dddot t}{\rho\dot t^4}+\frac{3\dot\rho\ddot t^2}{\rho\dot t^5}.$$

Substituting in this equation (2.4), (3.1), (3.2) and (2.6) written in the form

$$\frac{\delta^3x^\lambda}{dt^3}=-\frac{3\dot\rho}{\rho\dot t}\frac{\delta^2x^\lambda}{dt^2}-\frac{3}{\rho}\left(\frac{\ddot\rho}{\dot t^2}-\frac{\dot\rho\ddot t}{\dot t^3}\right)\frac{dx^\lambda}{dt}$$

$$-\frac{3}{\rho}\left(\frac{\dddot\rho}{\dot t^3}-\frac{3\ddot\rho\ddot t}{\dot t^4}-\frac{\dot\rho\dddot t}{\dot t^4}+\frac{3\dot\rho\ddot t^2}{\dot t^5}\right)x^\lambda,$$

we find, after a simple but a long calculation,

$$(3.3)\qquad \frac{da^0}{ds}+\Gamma^0_{jk}a^j\frac{d\xi^k}{ds}+\frac{d}{ds}\{t,s\}=0.$$

This is the first equation of (0.10).

Thus we are led to the conclusion that the conics and projective parameters on it defined by the equation (1.3) in D. van Dantzig's projective space coincide with those in E. Cartan's projective space with the components of the connection given by (1.8) and (1.9).