12. An Evaluation in the Theory of Multivalent Functions.

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1. In a previous paper we have considered a family $\mathfrak{F}_r$ of analytic functions which are regular and p-valent in the unit-circle $|z|<1$ and have the expansion of the type

(1) $$w(z) = z^p + a_{p+1} z^{p+1} + \ldots\ldots;$$

and proved that

$$|w(z)| \geqq \left(\frac{1}{1.0365\ldots}\right)^p \cdot \frac{|z|^p}{(1+|z|)^{2p}}$$

for $|z| \leqq x_0$ and

$$|w(z)| \geqq \left(\frac{1}{1.0604\ldots}\right)^p \cdot \frac{|z|^p}{(1+|z|)^{2p}}$$

for $x_0 \leqq |z| \leqq 1$, where $x_0 = 0.7389\ldots\ldots$

2. Here we want to ameliorate this result, and our new evaluation is as follows:

For $|z| \leqq x_1$

(2) $$|w(z)| \geqq \left(\frac{1}{1.00755\ldots}\right)^p \cdot \frac{|z|^p}{(1+|z|)^{2p}}$$

and for $x_1 \leqq |z| \leqq 1$

(3) $$|w(z)| \geqq \left(\frac{1}{1.03142\ldots}\right)^p \cdot \frac{|z|}{(1+|z|)^{2p}},$$

where $x_1 = 0.80458\ldots$

We will give here an outline of the demonstration of this result, and the detailed proof shall be given in another journal.

3. Our evaluation is based on the following theorems:

(I) $|a_{p+1}| \leqq 2p$[2]

(II) $$|w(z)|^{\frac{3}{2p}} \geqq \frac{|z|^{\frac{3}{2}}}{1+3|z|+2\sqrt{3}\,|z|\sqrt{\frac{|z|}{2}\log\frac{1+|z|}{1-|z|}}}, \text{ for } |z|<1.$$

The sketch of the proof of the inequality (II) shall be given in the following lines.

If we write

1) A. Kobori, Zur Theorie der mehrwertigen Funktionen. Japanese Journ. of Math. Vol. 19, 1947.

2) A. Kobori, loc. cit.

$$\left\{\frac{w(z)}{z^p}\right\}^{-\frac{3}{2p}} = 1 + b_1 z + b_2 z^2 + \dots + b_n z^n + \dots,$$

then we have, by the theorem of the previous paper,[1]

$$(4) \qquad \sum_{\nu=2}^{\infty}(2\nu - 3) \mid b_\nu \mid^2 \leqq 12,$$

and observing that

$$b_1 = -\frac{3}{2p} a_{p+1}$$

we have

$$\left|\left\{\frac{w(z)}{z^p}\right\}^{-\frac{3}{2p}} + \frac{3}{2p} a_{p+1} z - 1\right| = \left|\sum_{\nu=2}^{\infty} b_\nu z^\nu\right|$$

$$\leqq \sum_{\nu=2}^{\infty} \mid b_\nu \mid . \mid z \mid^\nu$$

$$= \sum_{\nu=2}^{\infty} \sqrt{2\nu - 3} \mid b_\nu \mid . \frac{\mid z \mid^\nu}{\sqrt{2\nu-3}}$$

$$\leqq \left\{\sum_{\nu=2}^{\infty} (2\nu-3) \mid b_\nu \mid^2\right\}^{\frac{1}{2}} . \left\{\sum_{\nu=2}^{\infty} \frac{\mid z \mid^{2\nu}}{2\nu-3}\right\}^{\frac{1}{2}}$$

By (4) and by an easy calculation[2] we obtain, for $\mid z \mid < 1$,

$$\left|\left\{\frac{w(z)}{z^p}\right\}^{-\frac{3}{3p}} + \frac{3}{2p} a_{p+1} z - 1\right| \leqq 2\sqrt{3} \mid z \mid \sqrt{\frac{\mid \mid}{2} \log \frac{1+\mid z \mid}{1-\mid z \mid}}$$

from which, taking in consideration the inequality (I), we can derive the inequality (II).

4. To attain our main object, therefore, it is sufficient to evaluate the real function

$$(5) \qquad \varphi(x) \equiv \frac{x^{\frac{2}{3}}}{1+3x+2\sqrt{3}x\sqrt{\frac{x}{2}\log\frac{1+x}{1-x}}},$$

and we have proved the following:

(III) *The real function* (5) *has, in the interval* $0 < x < 1$, *the greatest value for* $x = x_1 = 0.80458\dots$

(IV) *In the interval* $0 \leqq x \leqq x_1$, *we have*

$$1 + 3x + 2\sqrt{3}x\sqrt{\frac{x}{2}\log\frac{1+x}{1-x}} \leqq \bar{x}_0 (1+x)^3,$$

where $x_0 = 1.02913\dots$

1) A. Kobori, loc. cit.

2) For $\mid z \mid < 1$ we have

$$\sum_{\nu=2}^{\infty} \frac{\mid z \mid^{2\nu}}{2\nu-3} = \frac{\mid z \mid^3}{2} \log \frac{1+\mid z \mid}{1-\mid z \mid}$$

On use of these results there now follows:

$$| w(z) | \geqq \{\varphi(x_1)\}^{\frac{2p}{3}}, \text{ for } x_1 \leqq | z | \leqq 1,$$

and

$$| w(z) | \geqq \left(\frac{1}{x_0}\right)^{\frac{2}{3}p} \cdot \frac{x^p}{(1+x)^{2p}}, \text{ for } | z | \leqq x \leqq x_1.$$

From the former we can easily derive the second part of our theorem enun ciated in the paragraph 2, and the latter is nothing but the first part of the same theorem.