36. *A Theorem on the Poisson Integral.*

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1. We will prove the following theorem,

Theorem. Let $u(z)$ ($z = re^{i\theta}$) be a harmonic function in the unit circle $|z| < 1$ and be expressed by a Poisson integral:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi}) \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\varphi, \qquad (1)$$

where $u(e^{i\theta})$ is integrable in Lebesgue's sense, and G be any simply connected domain in $|z| < 1$.

When we map G conformally on the unit circle $|x| < 1$, $u(z)$ becomes a harmonic function $v(x)$ in $|x| < 1$.

Then $v(x)$ can be expressed by a Poisson integral of the form (1) in $|x| < 1$.

Prof. Tsuji proved this theorem in the special case in which G is bounded by a finite number of analytic curves C_i ($i = 1, \dots, k$) in $|z| < 1$ and a certain number of circular arcs on the unit circle $|z| = 1$, and the angles between any two adjoining C_i are different from zero and the angles which C_i makes with the unit circle are different from zero and π, so that C_i does not touch the unit circle.[1]

2. *Proof.* We write $u(z)$ in the form:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\{|u(e^{i\varphi})| + u(e^{i\varphi})\} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\varphi$$

$$- \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\{|u(e^{i\varphi})| - u(e^{i\varphi})\} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\varphi. \qquad (2)$$

Since both $\{|u(e^{i\theta})| + u(e^{i\theta})\}$ and $\{|u(e^{i\theta})| - u(e^{i\theta})\}$ are positive and integrable in Lebesgue's sense, $u(z)$ can be expressed by a difference of two positive harmonic functions of the form (1), so that to prove our theorem, it suffices to prove for a positive harmonic function of the form (1), where $u(e^{i\varphi}) \geqq 0$.

We take a sequence of positive numbers, such that

$$0 < M_1 < M_2 < \cdots\cdots < M_n \to \infty$$

and define $u_n(e^{i\theta})$ as follows;

$$u_n(e^{i\theta}) = u(e^{i\theta}) \qquad \text{when} \quad M_n \geqq u(e^{i\theta}),$$

$$u_n(e^{i\theta}) = M_n \qquad \text{when} \quad u(e^{i\theta}) > M_n,$$

(1) M. Tsuji, Theorems concerning Poisson integrals. Jap. Journ. Math. 7 (1930), 227—253.

so that $$0 \leqq u_n(e^{i\theta}) \leqq M_n \qquad \text{for } 0 \leqq \theta \leqq 2\pi. \tag{3}$$
We put
$$u_n(z) = \frac{1}{2\pi}\int_0^{2\pi} u_n(e^{i\varphi})\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\varphi,$$
then by (3)
$$u_n(z) \leqq \frac{1}{2\pi}\int_0^{2\pi} M_n \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\varphi = M_n$$
in $|z|<1$.
Since $$u_1(e^{i\theta}) \leqq u_2(e^{i\theta}) \leqq \cdots\cdots \leqq u_n(e^{i\theta}) \leqq \cdots\cdots,$$
we have
$$u_1(z) \leqq u_2(z) \leqq \cdots\cdots \leqq u_n(z) \leqq \cdots\cdots, \tag{4}$$
and by Lebesgue's theorem
$$\lim_{n\to\infty} u_n(z) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(e^{i\varphi})\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty} u_n(e^{i\varphi})\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\varphi$$
$$= u(z). \tag{5}$$

By the conformal mapping of G on $|x|<1$, $u_n(z)$ becomes a bounded harmonic function $v_n(x)$ in $|x|<1$, so that $v_n(x)$ can be expressed by
$$v_n(x) = \frac{1}{2\pi}\int_0^{2\pi} v_n(e^{i\varphi})\frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\,d\varphi \tag{6}$$
where $x=\rho e^{i\psi}$. From (4) and (5) we have
$$v_1(x) \leqq v_2(x) \leqq \cdots\cdots \leqq v_n(x) \leqq \cdots\cdots \tag{7}$$
and
$$\lim_{n\to\infty} v_n(x) = v(x). \tag{8}$$

By Fatou's theorem, $v_n(x)$ tends to $v_n(e^{i\psi})$ almost everywhere when $\rho \to 1$. Let e_n be a set on $|x|=1$ where $\lim_{\rho\to 1} v_n(\rho e^{i\psi})$ does not exist and put $e=\sum_{n=1}^{\infty} e_n$, $E=(0, 2\pi)-e$. Then
$$m\,e = 0 \tag{9}$$
because $m\,e_n = 0\,(n=1,2,\ldots\ldots)$, and on E $\lim_{\rho\to 1} v_n(\rho e^{i\psi}) = v_n(e^{i\psi})$ exists for all n. Therefore on E by (7)
$$v_n(e^{i\psi}) = \lim_{\rho\to 1} v_n(\rho e^{i\psi}) \leqq \lim_{\rho\to 1} v_{n+1}(\rho e^{i\psi}) = v_{n+1}(e^{i\psi}).$$
Hence on E
$$v_1(e^{i\psi}) \leqq v_2(e^{i\psi}) \leqq \cdots\cdots \leqq v_n(e^{i\psi}) \cdots\cdots,$$
and if we define
$$\overline{v}_n(e^{i\psi}) = v_n'(e^{i\psi}) \qquad \text{on } E,$$
$$\overline{v}_n(e^{i\psi}) = 0 \qquad \text{on } e,$$
then
$$\overline{v}_1(e^{i\psi}) \leqq \overline{v}_2(e^{i\psi}) \leqq \cdots\cdots \leqq \overline{v}_n(e^{i\psi}) \leqq \cdots\cdots \tag{10}$$

and by (9)

$$v_n(x) = \frac{1}{2\pi}\int_0^{2\pi} v_n(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\, d\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \bar{v}_n(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\, d\varphi. \tag{11}$$

From (7), (10), (11) and by Lebesgue's theorem

$$v(x) = \lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \bar{v}_n(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\, d\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty} \bar{v}_n(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\, d\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\, d\varphi,$$

where we put

$$\lim_{n\to\infty} \bar{v}_n(e^{i\varphi}) = v(e^{i\varphi})$$

which is integrable in Lebesgue's sense, because

$$v(0) = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\varphi})\, d\varphi < \infty.$$

That is,

$$v(x) = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\varphi)+\rho^2}\, d\varphi.$$

Remark. Let $z = f(x)$ be a single-valued regular function in $|x| < 1$ and suppose $|f(x)| < 1$. By this transformation $u(z)$ becomes a harmonic function $v(x)$ in $|x| < 1$. Then similarly as above, we can prove that $v(x)$ is expressed by a Poisson integral of the form (1) in $|x| < 1$. Therefore G is not necessarily a plane simply connected domain.