12. On the Cartan Decomposition of a Lie Algebra.

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Let $\mathfrak{L}$ be a Lie algebra over the field of complex numbers, $\mathfrak{H}$ and $\mathfrak{H}'$ maximal nilpotent subalgebras of $\mathfrak{L}$ containing regular elements. E. Cartan has shown for semi-simple $\mathfrak{L}$ that there exists an inner automorphism A such that $\mathfrak{H}'=A\mathfrak{H}$[1]. In this note we shall show that this theorem is valid for any, not necessarily semi-simple, Lie algebra. From this we see easily that the decomposition of a Lie algebra into the eigen-spaces of a maximal nilpotent subalgebra containing a regular element (Cartan decomposition) is unique up to inner automorphisms of $\mathfrak{L}$.

Let $\mathfrak{G}$ be the Lie group which corresponds to $\mathfrak{L}$. To every element a of $\mathfrak{L}$ corresponds a one-parameter subgroup $g(t)$ of $\mathfrak{G}$ and a is the tangent vector at the unit element to the differentiable curve $g(t)$. Extending to general Lie groups a notion familiar for matrices, we shgll denote by *exp ta* this one-parameter subgroup $g(t)$ and by *exp a* the point of parameter 1 on this curve. Further $exp\,\mathfrak{H}$ means the (local) subgroup of $\mathfrak{G}$ which corresponds to a subalgebra $\mathfrak{H}$ of $\mathfrak{L}$. If we transform the elements of the group *exp ta* by a fixed element g, we obtain a new one-parameter subgroup *exp ta'*; the mapping $a\to A_g a=a'$ is an inner automorphism of $\mathfrak{L}$ generated by g. The mapping $x\to D_a x=[a, x]$, with a fixed, is a linear operation in $\mathfrak{L}$, which is called inner derivation of $\mathfrak{L}$. Suppose that $g=exp\,a$ and g is sufficiently near to the unit element, then $A_g=exp\,D_a$. Let us decompose $\mathfrak{L}$ by A_g into eigen-spaces:

$$\mathfrak{L}=\mathfrak{L}_1+\mathfrak{L}_\rho+\mathfrak{L}_\sigma+\ldots\ldots$$

where $\mathfrak{L}_1$, $\mathfrak{L}_\rho$, ... are the eigen-spaces for the characteristic roots, 1, ρ, ... of A_g. Here $\mathfrak{L}_1$ is a subalgebra of $\mathfrak{L}$[2].

Lemma[3]. The systems $u^{-1}g\,exp\,\mathfrak{L}_1 u$, where u runs over a neighbourhood of the unit element, contain a neighbourhood of the element g in $\mathfrak{G}$.

Proof. Let $a_1, a_2, \ldots a_s$ be a basis of the subalgebra $\mathfrak{L}_1$ and $a_{s+1}, \ldots, a_r$ a basis of $\mathfrak{L}_\rho+\mathfrak{L}_\sigma+\ldots$. Then the (local) subgroup $exp\,\mathfrak{L}_1$ is composed of all elements of the forms $exp(t_1a_1+\ldots+t_sa_s)$, where the parameters t_i are sufficiently near to zero. To prove our Lemma, it is sufficient to show that the set of elements

1) E. Cartain, Le principe de dualité et la théorie des groupes simples et seme-simples (Bull. Sc. math. t. 49, 1925). Gantmacher has given a proof in a somewhat general form. F. Gantmacher, Canonical representations of automorphisms of a complex semi-simple Lie group, (Recueil mathématique, 5(47), 1939).

2) See Gantmacher, l. c. P. 107: If g is sufficiently near to the unit element then $\mathfrak{L}_1\neq 0$.

3) Cf. Gantmacher, l. c. P. 113.

$$g^{-1}exp(-(t_{s+1}a_{s+1}+\dots+t_ra_r))g exp(t_1a_1+\dots+t_sa_s)exp(t_{s+1}a_{s+1}+\dots+t_ra_r)$$
$$=exp(p_1a_1+p_2a_2+\dots+p_ra_r),$$

where $p_i=p_i(t_1, \dots, t_r)$ are analytic functions of t_k, cover a neighbourhood of the unit element, when t_k run independently over a neighbourhood of zero. To see this it suffices to show that the Jacobian

$$\frac{\partial(p_1\dots p_r)}{\partial(t_1\dots t_r)}$$

is different from zero for $t_1=t_2=\dots=t_r=0$.
Let $t_i=0$ for $i\neq j$, $1\leqq j\leqq s$. Then

$$exp(t_ja_j)=exp\sum_{i=1}^{r}p_i(0\dots t_j\dots 0)a_i.$$

Hence $p_i(0\dots t_j\dots 0)=\delta_{ij}t_j$ and $\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0}=\delta_{ij}$, for $1\leqq j\leqq s$.
Now, let $t_i=0$ for $i\neq j$, $j>s$. Then

$$g^{-1}exp(-t_ja_j)g\cdot exp(t_ja_j)=exp(-t_jA_ga_j)exp(t_ja_j)$$
$$=exp(\sum_{i=1}^{r}p_i(0\dots t_j\dots 0)a_i).$$

From this we obtain the equations

$$(1-A_g)a_j=\sum_{i=1}^{r}\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0}a_i, \text{ for } j>s.$$

Since the linear operation $1-A_g$ transforms the space $\mathfrak{L}_\rho+\mathfrak{L}_\sigma+\dots$ into itself and is non singular on this space, $\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0}=0$ for $1\leqq i\leqq s$, $j>s$, and the matrix

$$\left(\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0}\right)_{s+1\leq i,\, j\leq r}$$

is non-singular.

Thus
$$\frac{\partial(p_1\dots p_r)}{\partial(t_1-t_r)}_{t=0}\neq 0.$$

Now let $a_1, a_2 \dots, a_r$ be a basis of $\mathfrak{L}$, $\xi_1a_1+\xi_2a_2+\dots+\xi_ra_r$ a general element of $\mathfrak{L}$ ($\xi_1, \dots, \xi_r$ are variables) and let

$$|tE-(\xi_1D_{a_1}+\dots+\xi_rD_{a_r})|=t^r-\psi_1(\xi)t^{r-1}+\dots\pm\psi_{r-l}(\xi)t^l$$

be the characteristic equation of $\mathfrak{L}$.

An element $a=\sum_{i=1}^{r}\lambda_ia_i$ of $\mathfrak{L}$ is called regular, if $\psi_{r-l}(\lambda)\neq 0$.

The totality of regular elements is an ophn set in r dimensional complex vector space $\mathfrak{L}$ and singular elements form an algebraic manifold of at most $r-1$ complex dimensions. Hence the set of all regular elements is connected.

Let $a=\sum_{i=1}^{r}\lambda_ia_i$ and $b=\sum_{i=1}^{r}\mu_ia_i$ be two regular elements, $\mathfrak{H}_a$ and $\mathfrak{H}_b$ the maximal nilpotent subalgebras of $\mathfrak{L}$ containing a and b respectively. First let the parameters (μ_i) be sufficiently near to (λ_i). We choose a sufficiently small positive number ξ such that $D_{\xi a}=\xi D_a$ has no characteristic roots of the form $2\pi\sqrt{-1}m$, where m denotes integer $\neq 0$. Let $g=exp(\xi a)$ and $\mathfrak{L}_1$ be the eigen-space for the characteristic root 1 of the inner automorphism A_g. Since $\mathfrak{H}_a$ is the eigen-space for the chaarcteristic root 0 of the inner derivation $D_{\xi a}$, $A_g=exp\,D_{\xi a}$ and moreover, $D_{\xi a}$ has no characteristic root of the form $2\pi\sqrt{-1}m$, we have $\mathfrak{L}_1=\mathfrak{H}_a$. As $exp\xi b$ is sufficiently near to g, there exists, by the above lemma, an element $u\epsilon\mathfrak{G}$ such that $exp\xi b\epsilon u^{-1}g\,exp\mathfrak{H}_a u$. But since g is contained in $exp\,\mathfrak{H}_a$, we have $g\,exp\,\mathfrak{H}_a\leqq exp\,\mathfrak{H}_a$ Hence $exp\xi b\epsilon u^{-1}exp\,\mathfrak{H}_a u=exp\,A_u\mathfrak{H}_a$. Thus there exists an element $c\epsilon\mathfrak{H}_a$, which is also regular such that $\xi b=A_u c$. Then we obtain

$$\mathfrak{H}_b=\mathfrak{H}_{\xi b}=\mathfrak{H}A_{uc}=A_u\mathfrak{H}_c,$$

and since $c\epsilon\mathfrak{H}_a$, $\mathfrak{H}_c=\mathfrak{H}_a$. Thus $\mathfrak{H}_b=A_u\mathfrak{H}_a$. Now let a and b be arbitrary regular element. Since the set of all regular elements is connected, we see by continuity that there exists an inner automorphism A such that $A\mathfrak{H}_a=\mathfrak{H}_a$.

Thus we have proved the following.

Theorem. Let $\mathfrak{H}$and $\mathfrak{H}'$ be two maximal nilpotent subalgebras containing regular elements of a Lie algebra $\mathfrak{L}$ over the field of complex numbers. Then there exists an inner automorphism A such that $\mathfrak{H}'=A\mathfrak{H}$. The Cartan decompositionof $\mathfrak{L}$ is unique up to inner automorphisms of $\mathfrak{L}$.

Added in proof (May 2, 1950).

After the present note was submitted to the Proc. of Acad. of Japan, I was made aware through Mathematical Reviews that the result in the present note had been already proved by C. Chevalley in his paper, " An algebraic proof of a property of Lie groups," Amer. J. Math. v. 63 (1941). But I assume that my approach is different from his.