11. *Note on the Replicas of Matrices.*

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The concept of the replicas of matrices was introduced by C. Chevalley[1]. and very interesting application of it to the study of algebraic Lie groups was given in a joint paper by himself and H. F. Tuan[2]. Chevalley determined the replicas of matrices over a field of characteristic zero and H. F. Tuan[3] gave an elementary proof to the same result and in fact in a somewhat general form. In the present note[4] we shall prove Chevalley's results in a somewhat different way and obtain some properties of the replicas which shall be used in a forthcoming paper[5].

§ 1. A replica B of a matrix A, of degree n with coefficients in a field K[6], is any matrix B which admits as its invariants all the tensor invariants of A, where A is meant to be the symbol of in finitesimal. not a finite transformation. Let $\mathfrak{M}$ be the vector space on which our matrices operate, $\mathfrak{M}^*$ the space of contravariant vectors, and $\mathfrak{T}_{rs}$ the space of r times contravariant and s times covariant tensors. We denote by A^* and A_{rs} the matrices of linear transformations which are induced by A in $\mathfrak{M}^*$ and $\mathfrak{T}_{rs}$ respectively.

Lemma 1. Any matrix A may be represented in the form

$$(1) \qquad A=A_0+A', \quad A_0A'=A'A_0$$

where A is a nilpotent matrix and A' is a matrix with simple elementary divisors. If A is given, A and A' are determined uniquely.

Proof. $\mathfrak{M}$ is the direct sum of the eigen-spaces: $\mathfrak{M}=\sum_{\lambda}\mathfrak{M}_\lambda$, where $\mathfrak{M}_\lambda$ denotes the eigen-space for a characteristic root λ of A. We define the matrix (or linear transformation) A' by the equations

$$A'x=\lambda x, \quad \text{for} \quad x\epsilon\mathfrak{M}_\lambda$$

A' commutes with A and, if we put $A_0=A-A'$. A_0 is nilpotent and commutes with A'. The uniqueness of this representation can be proved easily from the commutability of A and A'.

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1) C. Chevalley, On a kind of new relationship between matrices, Amer, J. Math. Vol. 65 (1943). I have not yet access to this paper and the results obtained in this note may perhaps have much contact with Chevalley's paper.

2) C. Chevalley and H. F. Tuan, On algebraic Lie algebras, Proc. Nat. Acad. Sci. U. S. A. (1946).

3) H. F. Tuan, A note on the replicas of nilpotent matrices, Bull. Amer. Math. Soc. (1945).

4) I express my hearty thanks to M. Gotô for his kind remarks during the preparation of this note.

5) Y. Matsushima, On algebraic Lie groups and algebras, Journ. of the Math. Soc. of Japan, Vol. 1 No. 1 (1948).

6) In the following we assume for simplicity that the field K is algebraically closed.

Lemma 2. The matrices A_0 and A' in Lemma 1 are the replicas of A.

Proof. Let $A=A_0+A'$ be the representation of Lemma 1. Then $A_{rs}=(A_0)_{rs}+A_{rs}'$, $(A_0)_{rs}(A')_{rs}=(A')_{rs}(A_0)_{rs}$ and $(A_0)_{rs}$ is a nilpotent matrix and A_{rs}' is a matrix with simple elementary divisors. We denote by $\mathfrak{T}_\kappa$ the eigen-space of $\mathfrak{T}=\mathfrak{T}_{rs}$ for a characteristic root κ of A_{rs}. Then, if $F_\kappa\in\mathfrak{T}_\kappa$, we have $A_{rs}'F_\kappa=\kappa F_\kappa$ (see, Lemma 1). Let $F\in\mathfrak{T}_{rs}$ and $A_{rs}F=0$. Since F belongs to $\mathfrak{T}_0$, we have $A_{rs}'F=0$. This shows that A' is a replica of A and therefore $A_0=A-A'$ is also a replica of A.

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct characteristic roots of A and l the maximal number of these characteristic roots which are linearly independent with respect to the prime field P in K. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_l$ are linearly independent. Then we have

$$\lambda_i=\sum_{j=1}^{l} r_{ij}\lambda_j \quad r_{ij}\in P \quad (i=1, 2, \dots, k) \tag{2}$$

Further, let E_i be the matrix of projection of $\mathfrak{M}$ on $\mathfrak{M}_{\lambda_i}$, *i.e.* if $x=\sum_{i=1}^{k} x_i$, $x_i\in\mathfrak{M}_{\lambda_i}$, is a vector of $\mathfrak{M}$, we define E_i by the equation

$$E_i x=x_i.$$

From the definition of A', we have $A'=\sum_{i=1}^{k}\lambda_i E_i$

If we put

$$A_j=\sum_{i=1}^{k} r_{ij}E_i \quad (j=1, 2, \dots l) \tag{3}$$

we can represent A in the form

$$A=A_0+\lambda_1A_1+\lambda_2A_2+\dots+\lambda_lA_l \tag{4}$$

and A_j $(j=1, \dots, l)$ are matrices with simple elementary divisors and their characteristic roots belong to the prime field P.

Lemma 3. The matrices $A_1, A_2 \dots, A_l$ difined by (3) are the replicas of A.

Proof. Let F be a tensor of $\mathfrak{T}_{rs}$ and let $A_{rs}F=0$. F belongs to the eigen-space $\mathfrak{T}_0$ for the characteristic root zero of A_{rs}. If we denote by $\mathfrak{M}^*_{-\lambda_i}$ the eigen-space of $\mathfrak{M}^*$ for a characteristic root $-\lambda_i$ of A^*, we have

$$\mathfrak{T}_0=\sum\mathfrak{M}^*_{-\lambda_{i_1}}\times\mathfrak{M}^*_{-\lambda_{i_2}}\times\dots\times\mathfrak{M}^*_{-\lambda_{i_r}}\times\mathfrak{M}_{\lambda_{j_1}}\times\dots\times\mathfrak{M}_{\lambda_{j_s}}$$

where $\times$ mean direct (Kronecker) product and the summation is extended over all combinations $(-\lambda_{i_1}, \dots -\lambda_{i_r}, \lambda_{j_1}, \dots, \lambda_{j_s})$ such that

$$-\lambda_{i_1}-\dots-\lambda_{i_r}+\dots+\lambda_{j_s}=0.$$

Clearly it is sufficient to prove that $(A_t)_{rs}F=0$, for

$$F\in\mathfrak{M}^*_{-\lambda_{i_1}}\times\dots\times\mathfrak{M}_{-\lambda_{i_r}}\times\mathfrak{M}^*_{\lambda_{j_1}}\times\dots\times\mathfrak{M}_{\lambda_{j_s}}.$$

If we operate A_t on F, F is multiplied by

$$-r_{i_1,t}-\dots-r_{i_rt}+r_{j_1t}+\dots+r_{j_st}.$$

But since

$$-\lambda_{i_1}-\ldots-\lambda_{i_r}+\lambda_{j_1}+\ldots+\lambda_{j_s}=\sum_{t=1}^{l}(-r_{i_1t}-\ldots-r_{i_rt}+r_{j_1t}+\ldots\ldots r_{j_st})\lambda_t=0,$$

we have $-r_{i_1t}-\ldots-r_{i_rt}+r_{j_1t}+\ldots+r_{j_st}=0.$ $(t=1,\ 2,\ \ldots,\ l)$

Consequently we have $(A_t)_{rs}F=0$.

The matrix A can be transformed into the following form:

(5)

$$A=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix},\ \text{where}\ A_t=\begin{pmatrix} \boxed{\begin{matrix}\lambda_t & & \\ & \ddots & \\ & & \lambda_t\end{matrix}} & & & \\ & \boxed{\begin{matrix}\lambda_t & & \\ 1 & & \\ & 1 & \lambda_t\end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix}\lambda_t & & \\ 1 & & \\ & 1 & \lambda_t\end{matrix}} \end{pmatrix}$$

Let $x^{\alpha i}(\alpha=0,1,\ldots\omega,\ \omega+1$ being the number of blockes in A_j; $i=0,1,\ldots k_\alpha)$ be the contravariant variables such that

$$\begin{cases} A\ x^{0i}=\lambda_j x^{0i} & (i=0,\ 1,\ \ldots k_0) \\ A\ x^{\alpha 0}=\lambda_j x^{\alpha 0} & (\alpha=1,\ 2,\ \ldots,\ \omega) \\ A\ x^{\alpha i}=x^{\alpha i-1}+\lambda_j x^{\alpha i} & (\alpha=1,\ 2,\ \ldots,\ \omega;\ i=1,\ 2,\ \ldots k_\alpha) \end{cases}$$

and $y_{\alpha i}$ $(\alpha=0,1,\ldots,\omega;\ i=0,1,\ldots k_\alpha)$ be the covariant variables such that

$$\begin{cases} A\ y_{0i}=-\lambda_j y_{0i} & (i=0,\ 1,\ \ldots k_0) \\ A\ y_{\alpha i}=-\lambda_j y_{\alpha i}-y_{\alpha i+1} & (\alpha=1,\ 2,\ \ldots,\ \omega;\ i=0,\ 1,\ \ldots k_\alpha-1) \\ A\ y_{\alpha k_\alpha}=-\lambda_j y_{\alpha k_\alpha} & (\alpha=1,\ 2,\ \ldots\omega) \end{cases}$$

We put

$$(6)\qquad \begin{cases} F_i^k=y_{0i}x^{0k} \\ \overset{0}{F}{}_\alpha^\beta=y_{\alpha k_\alpha}x^{\beta 0}, \\ \overset{1}{F}{}_\alpha^\beta=y_{\alpha k_\alpha-1}x^{\beta 0}+y_{\alpha k_\alpha}x^{\beta 1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \overset{k'}{F}{}_\alpha^\beta=y_{\alpha k_\alpha-k'}x^{\beta 0}+y_{\alpha k_\alpha-k'+1}x^{\beta 1}+\ldots+y_{\alpha k_\alpha}x^{\beta k'}, \end{cases}$$

where $k'=\min(k_\alpha,\ k_\beta)$, $\alpha,\ \beta=1,\ 2,\ \ldots\omega$.

Then, these F's are invariant tensors of A in $\mathfrak{T}_{11}$[7], as we can verify easily. Let now B be any replica of A. Since the matrix B admits as its invariants all tensors from (6), we see that the matrix B must have in this coordinate system the form:

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7) These invariants were found by M. Gotô.

(7)

$$B=\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}, \text{ where } B_i=\begin{pmatrix} \boxed{\begin{matrix}\mu_i & & \\ & \diagdown & \\ & & \mu_i\end{matrix}} & & & \\ & \boxed{\begin{matrix}\mu_i & & \\ * & \diagdown & \\ & & \mu_i\end{matrix}} & & \\ & & \diagdown & \\ & & & \boxed{\begin{matrix}\mu_i & & \\ * & \diagdown & \\ & & \mu_i\end{matrix}} \end{pmatrix}$$

where $\mu_1, \mu_2, \ldots, \mu_k$ are characteristic roots of B.

Lemma 4. The linear relation in λ_i, $\sum_{i=1}^{k} r_i\lambda_i=0$, $r_i\epsilon P$, implies the same relation in μ_i : $\sum_{i=1}^{k} r_i\mu_i=0$.

Proof. First let P be of characteristic $p\neq 0$. We consider r_i as rational integers mod p. Since the matrices A and B have the forms (5) and (7), there exist covariant vectors $x_1, \ldots, x_k$ such that

$$Ax_i=\lambda_i x_i,\ Bx_i=\mu_i x_i,\ i=1,\ 2,\ \ldots k.$$

Put

$$F=\underbrace{x_1\ldots x_1}_{r_1 \text{ times}}\ \underbrace{x_2\ldots x_2}_{r_2 \text{ times}}\ \ldots\underbrace{x_k\ldots x_k}_{r_k \text{ times}}$$

F is a tensor invarinat of A, since, from the above relation, we have $A_{0s}F=0$, where $s=r_1+r_2+\ldots+r_k$. As B is a replica of A, F must be an invariana of B and this implies $\sum_{i=1}^{k} r_i\mu_i=0$. The proof for the case of characteristic zero runs analogously, if we choose, as we may, r_i as rational integers.

We represent B in the form :

$$B=B_0+B',\quad B_0B'=B'B_0.$$

where B is a nilpotent matrix and B' is a matrix with simple elementary divisors. We may prove the following.

Lemma 5. The matrix B_0 is a replica of the matrix A_0.

Proof. we take the basis $\{v_1\ldots v_m\ u_{11}\ldots u_{1n_1}\ldots u_{q1}\ldots u_{qn_q}\}$ of the eigen-space $\mathfrak{T}_\kappa$ of $\mathfrak{T}_{rs}$ for a characteristic root κ or A_{rs} such that

$$(8)\qquad \begin{cases} A_{rs}v_i=\kappa v_i & (i=1,\ 2,\ \ldots m) \\ A_{rs}u_{ij}=\kappa u_{ij}+u_{ij+1} & (j\neq n_i,\ i=1,\ 2,\ \ldots,\ q) \\ A_{rs}u_{in_i}=\kappa u_{in_i} & (i=1,\ 2,\ \ldots q) \end{cases}$$

We will show that $A_{rs}F=A_{rs}'F$, $F\epsilon\mathfrak{T}_{rs}$ implies $B_{rs}F=B_{rs}'F$. We represent F in the form $F=\underset{\kappa}{} F_\kappa$, $F_\kappa\epsilon\mathfrak{T}_\kappa$. Since

$$A_{rs}F=\underset{\kappa}{} A_{rs}F_\kappa=A_{rs}'F=\sum_{\kappa}\kappa F_\kappa$$

We have $$A_{rs}F_\kappa=\kappa F_\kappa.$$

From this we see easily that F_κ has the following form:

$$F_\kappa=\sum_{i=1}^{m}\alpha_i v_i+\sum_{i=1}^{q}\beta_i u_{in_i}$$

Since B_{rs} is evidently a replica of A_{rs}, we have from (5), (7) and (8) the following equations:

$$\begin{cases} B_{rs}v_i=\nu v_i & (i=1,\ 2,\ \ldots m) \\ B_{rs}u_{ij}=\nu u_{ij}+\gamma_{ij+1}u_{ij+1}+\ldots+\gamma_{in_i}u_{in_i} & (j\neq n_i,\ i=1,\ 2,\ \ldots q) \\ B_{rs}u_{in_i}=\nu u_{in_i} & (i=1,\ 2,\ \ldots q) \\ B_{rs}'F_\kappa=\nu F_\kappa,\ F_\kappa\epsilon\mathfrak{T}_\kappa. \end{cases}$$

Hence we have

$$B_{rs}F_\kappa=\nu F_\kappa=B_{rs}'F_\kappa \text{ and } B_{rs}F_\kappa=B_{rs}'F_\kappa,$$

which shows that B is a replica of A_0.

From these lemmas we may prove the following

Theorem. A necessary and suffiicient condition that a matrix B is a replica of the matrix A is that B is of the form,

$$B=\tilde{A}_0+\mu_1A_1+\mu_2A_2+\ldots+\mu_lA_l \tag{9}$$

where $\tilde{A}_0$ is an arbitrary replica of the nilpotent matrix A_0 and μ_i $(i=1,\ 2,\ \ldots l)$ are arbitrry elements of the field K.

Proof. The suffiiciency may be seen from the Lemma 2 and 3. Let, conversely, B be a replica of A. We represent B, as in Lemma 1, in the form $B=B_0+B'$.

Then we have from Lemma 5 $B_0=\tilde{A}_0$, where $\tilde{A}_0$ is a replics of A_0. The relations (2) and Lemma 4 imply the relations,

$$\mu_i=\sum_{j=1}^{l}r_{ij}\mu_j,\ r_{ij}\epsilon P \qquad (i=1,\ 2,\ \ldots k) \tag{10}$$

where μ_i are defined in (7). From the representation

$$B'=\mu_1E_1+\mu_2E_2+\ldots+\mu_kE_k$$

and (10), we have

$$B'=\sum_{i=1}^{k}\sum_{j=1}^{l}r_{ij}\mu_jE_i=\sum_{j=1}^{l}\mu_j\sum_{i=1}^{k}r_{ij}E_i=\sum_{j=1}^{l}\mu_jA_j.$$

Consequently we have

$$B=\tilde{A}_0+\sum_{i=1}^{i}\mu_iA_i$$

Now the replicas of nilpotent matrices have been determined by Chevalley and H. E. Tuan and have the following forms:

$$\tilde{A}=\begin{cases}\nu A_0, \text{ if the characteristic of the field } K \text{ is zero,} \\ \sum_i \nu_i A_0^{p^i}, \text{ if the characteristic of the field } K \text{ is } p\neq 0.\end{cases}$$

§ 2. Let $\mathfrak{M}$ be the direct of some tensor spaces and $\tilde{\mathfrak{N}}$ a subspace of $\mathfrak{M}$ which is invariant under A.

We denote by $\tilde{A}$ the matrix of linear transformation which is induced by A in $\tilde{\mathfrak{N}}$. we shall consider the connection between the replicas of A and those of $\bar{A}$.

Lemma 6. Let $\mathfrak{N}$ be the subspace of $\mathfrak{M}$ which is invariant under A and $\bar{A}$ the matrix of linear transformation which is induced by A in $\mathfrak{N}$. Then $\mathfrak{N}$ is invariant under all replicas of A and the matrices of linear transformations which are induced by the replicas of A in $\mathfrak{N}$ are replicas of the matrix $\bar{A}$, and conversely any replica of $\bar{A}$ is induced by a replica of A.

Proof. Let m be the maximal number of the characteristic roots of $\bar{A}$ which are linearly independent with respect to P.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{l+1}, \ldots, \lambda_q$ be the distinct characteristic roots of $\bar{A}$ and let $\lambda_1, \ldots, \lambda_m$ be linesrly independent.

Let l and $\lambda_1 \ldots, \lambda_m, \ldots, \lambda_l, \lambda_{l+1}, \ldots \lambda_q, \ldots \lambda_k$ be defined for A analogously.

We denote by $\mathfrak{N}_{\lambda_i}$ the eigen-space of $\mathfrak{N}$ for a characteristic root λ_i of $\bar{A}$ and by E_i' the matrix of projection of $\mathfrak{N}$ on $\mathfrak{N}_{\lambda_i}$. Since $\mathfrak{M}_{\lambda_i} \supseteq \mathfrak{N}_{\lambda_i}$, $\mathfrak{N}$ is invariant under E_i and $\bar{E}_i = E_i'$ or $\bar{E}_i = 0$ according as λ_i is a characteristic root of $\bar{A}$ or not. Hence, by (3) and Theorem of § 1, $\mathfrak{N}$ is invariaht under all replicas of A.

Let
$$\lambda_i = \sum_{j=1}^{l} r_{ij} \lambda^j \qquad (i=1, 2, \ldots, k),$$
where r_{ij} are elements of P and where $r_{ij}=0$ for
$$l+1 \leqq i \leqq q, \quad m+1 \leqq j \leqq l.$$

If we put
$$A_j' = \sum_{i=1}^{k} r_{ij} E_i' \quad (j=1, 2, \ldots, m),$$
where $E_i'=0$ if λ_i is not a characteristic root of $\bar{A}$, then we may represent $\bar{A}$ in the form
$$\bar{A} = A_0' + \lambda_1 A_1' + \lambda_2 A_2' + \ldots + \lambda_m A_m' \tag{12}$$
and every replica of $\bar{A}$ is represented in the form,
$$\sum_i \nu_i (A_0')^{p^i} + \sum_{i=1}^{m} \mu_i A_i'. \tag{13}$$
But we verify easily that
$$\bar{A}_j = A_j' \text{ for } 1 \leqq j \leqq m \text{ and } \bar{A}_j = 0 \text{ for } m+1 \leqq j \leqq l.$$
From these relations, we get $\bar{A}_0 = A_0'$ and $\bar{A}_0^{p^i} = (A_0')^{p^i}$.
Let B be a replica of A. Then B is represented in the form.
$$B = \sum_i \nu_i A_0^{p^i} + \sum_{i=1}^{l} \mu_i A_i.$$
Then $\bar{B} = \sum_i \nu_i (A_0')^{p^i} + \sum_{i=1}^{m} \mu_i A_i'$ is a replica of $\bar{A}$. Conversely we see easily that any replica of $\bar{A}$ is induced by a replica of A.

Lemma 7. Let

$$\tilde{A}=\begin{pmatrix} A & & & \\ & A_{r_1 s_1} & & \\ & & \ddots & \\ & & & A_{r_t s_t} \end{pmatrix}$$

If B is a replica of A, then the matrix

$$\tilde{B}=\begin{pmatrix} B & & & \\ & B_{r_1 s_1} & & \\ & & \ddots & \\ & & & B_{r_t s_t} \end{pmatrix}$$

is a replica of the matrix $\tilde{A}$. Conversely any replica of $\tilde{A}$ is of the form $\tilde{B}$, where B is a replica of A.

Proof. For simplicity, we prove this lemma in the case $\tilde{A}=\begin{pmatrix} A & \\ & A_{rs}\end{pmatrix}$. If we represent A as in (4). then

$$\tilde{A}=\begin{pmatrix} A_0 & \\ & (A_0)_{rs}\end{pmatrix}+\lambda_1\begin{pmatrix} A_1 & \\ & (A_1)_{rs}\end{pmatrix}+\dots+\lambda_l\begin{pmatrix} A_l & \\ & (A_l)_{rs}\end{pmatrix}$$
$$=\tilde{A}_0+\lambda_1\tilde{A}_1+\dots+\lambda_l\tilde{A}_l,$$

where $$\tilde{A}_l=\begin{pmatrix} A_l & \\ & (A_l)_{rs}\end{pmatrix}$$

The linear space on which $\tilde{A}$ operates is $\tilde{\mathfrak{M}}=\mathfrak{M}+\mathfrak{T}_{rs}$ and the characteristic roots of $\tilde{A}$ are $\lambda_1, \lambda_2, \dots, \lambda_k$ and

$$\kappa=-\lambda_{i_1}-\dots-\lambda_{i_r}+\lambda_{j_1}+\dots+\lambda_{j_s}=\sum_{t=1}^{l}(-r_{i_1 t}-\dots-r_{i_r t}+r_{j_1 t}+\dots+r_{j_s t})\lambda_t,$$

where $i_1, \cdots, i_r, j_1, \cdots, j_s$ are chosen arbitrarily from $1, 2, \cdots. k$,
If $\kappa=-\lambda_{i_1}-\dots-\lambda_{i_r}+\lambda_{j_1}+\dots+\lambda_{j_s}=-\lambda_{p_1}-\dots-\lambda_{p_r}+\lambda_{q_1}+\dots+\lambda_{q_s}$, then $-r_{i_1 t}-\dots-r_{i_r}+r_{j_s t}+\dots+r_{j_s t}=-r_{p_1 t}-\dots-r_{p_r t}+r_{q_1 t}+\dots+r_{q_s t}$ for $t=1, \dots, l$. We denote by $\tilde{E}_i$ and $\tilde{E}_\kappa$ the matrices of projection of $\tilde{\mathfrak{M}}$ on the eigen-space for the characteristic root λ_i and κ of $\tilde{A}$ respectively. To prove the lemma, it is sufficient to show that

$$\tilde{A}_t=\sum_{i=1}^{k} r_{it}\tilde{E}_i+\sum_{\kappa}(-r_{i_1 t}-\dots-r_{i_r t}+r_{j_1 t}+\dots+r_{j_s t})\tilde{E}_\kappa \tag{14}$$

The eigen-space $\tilde{\mathfrak{M}}_\kappa$ $(\kappa\neq\lambda^i)$ is of the form

$$\tilde{\mathfrak{M}}_\kappa=\sum\mathfrak{M}^*_{-\lambda_{i_1}}\times\dots\times\mathfrak{M}^*_{-\lambda_{i_r}}\times\mathfrak{M}_{\lambda_{j_1}}\times\dots\times\mathfrak{M}_{\lambda_{j_s}},$$

where the summation is extended over all combinations $(-\lambda_{i_1}, \dots, -\lambda_{j_r}, \lambda_{j_1}, \dots, \lambda_{j_s})$ such that $-\lambda_{i_1}-\dots-\lambda_{i_r}+\dots+\lambda_{j_s}=\kappa$.
If we operate $\tilde{A}_t$ on the elements of $\tilde{\mathfrak{M}}_\kappa$, these are merely multiplied by $-r_{i_1 t}-\dots-r_{i_r t}+r_{j_1 t}+\dots+r_{j_s s}$, hence we get (14) on $\tilde{\mathfrak{M}}_\kappa$.

The eigen-space $\tilde{\mathfrak{M}}_{\lambda u}$ is of the form

$$\tilde{\mathfrak{M}}_{\lambda_u}=\mathfrak{M}_{\lambda_u} \text{ or } \tilde{\mathfrak{M}}_{\lambda_u}=\mathfrak{M}_{\lambda_u}+\sum\mathfrak{M}^*_{-\lambda_{i_1}}\times\dots\times\mathfrak{M}^*_{-\lambda_{i_r}}\times\mathfrak{M}_{\lambda_{i_1}}\times\dots\times\mathfrak{M}_{\lambda_{j_s}},$$

where the summation is extended over all combinations

$(-\lambda_{i_1}, \dots, -\lambda_{i_r}, \lambda_{j_1}, \dots, \lambda_{j_s})$ such that $-\lambda_{i_1}-\dots-\lambda_{i_r}+\lambda_{j_1}+\dots+\lambda_{j_s}=\lambda_u$.
Analogously we get (14) on $\tilde{\mathfrak{M}}_{\lambda_u}$. Hence (14) is valid on $\tilde{\mathfrak{M}}$.

From Lemma 6 and 7 we get the following result.

Let $\mathfrak{M}$ be the direct sum of some tensor spaces and $\tilde{\mathfrak{N}}$ a subspace of $\mathfrak{M}$ which is invariant under A. We denote by $\tilde{A}$ the matrix of linear transformation which is induced by A in $\tilde{\mathfrak{N}}$. Then $\tilde{\mathfrak{N}}$ is invariant under all replicas of A and the matrices of linear transformations which are induced by the replicas of A in $\tilde{\mathfrak{N}}$ are replicas of the matrix $\tilde{A}$ and conversely any replica of $\tilde{A}$ is induced by a replica of A in $\tilde{\mathfrak{N}}$.