10. *On the Replicas of Nilpotent Matrices.*

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§ 1. Recently[1], C. Chevalley and H. F. Tuan proved the following theorem:

THEOREM 1. *If Z is a nilpotent matrix over a field P, only replicas Z' of Z are the matrices of the from $Z'=q(Z)$, where $q(x)$ is a polynomial in P, and according as P is of characteristic* 0 *or $p\neq 0$*

$$(1) \qquad \begin{cases} q(x)=tx & t\in P, \\ or & \\ q(x)=q_0x+q_1x^p+q_2x^{p^2}+\cdots & q_i\in P. \end{cases}$$

In order to prove this, Tuan used the following Lemma 1 due to Chevalley and deduced it as a consequence of Lemma 2.

LEMMA 1. *If Z' is a replica of nilpotent Z, then Z' is a polynomial of Z with coefficients in P and without constant term.*

LEMMA 2.[2] *If Z and Z' are nilpotent matrices, and if $q(x)$ and $s(x)$ are polynomials in P without constant terms such that*

$$Z'=q(Z),$$
$$Z'_{02}=s(Z_{0,2}),\ or\ Z'_{1,1}=s(Z_{1,1}),$$

then $q(x)$ is of the form (1).

Here we shall give a direct simple proof of a lemma (Lemma 1′) stronger than Lemma 1 and prove the theorem under somewhat weaker conditions. Now we denote by $n(A_{m,n})$ the space of all tensor invariants of a matrix A in $\mathfrak{T}_{m,n}$. Then our lemma is

LEMMA 1′. *If Z is a nilpotent matrix and Z' a matrix of the same degree such that*

$$n(Z')\supset n(Z)$$
$$n(Z'_{0,2})\supset n(Z_{0,2}),\quad or\quad n(Z'_{1,1})\supset n(Z_{1,1}),$$

then Z' is a polynomial of Z without constant term.

PROOF. It is clearly surfficient to prove the case when the field P is algebraically closed and Z is of the so-called Jordan's normal form. Then Z may be represented by

$$x^S\to 0 \qquad S=01,\ \dots\ 0n,\ 10,\ \dots,\ m0$$
$$x^{ah}\to x^{a\ h-1} \qquad a=1,\ 2,\ \dots,\ m. \qquad h=1,\ 2,\ \dots\ k(a).$$

Now the tensors

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1) C. Chevalley, "A new kind of relationship between matrices," Amer. J. of Math., v. 65 (1943). H. F. Tuan, "A note on the replicas of nilpotent matrices," Bull. of Amer. Math. Soc. (1945). Cf. also C. Chevalley and H. F. Tuan, "On algebraic Lie algebras," Proc. Nat. Acad. Sci., U. S. A. (1946). We have had no access to the first paper, and the definitions and notations are borrowed from the latter two papers in the present note.

2) Tuan proved Lemma 2 under somewhat stronger conditions, but his proof is based essentially on the assumptions of our Lemma 2 alone.

$$(2) \qquad x^{01},\ \ldots,\ x^{0n},\ x^{10},\ \ldots x^{m0}$$

$$(3) \qquad \begin{cases} x_S y^T \qquad S,\ T=01,\ \ldots,\ 0n,\ 10,\ \ldots,\ m0 \\ \overset{1}{F}{}^{ab}=x^{a1}y^{b0}-x^{a0}y^{b1}, \\ \overset{2}{F}{}^{ab}=x^{a2}y^{b0}-x^{a1}y^{b1}+x^{a0}y^{b2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \overset{k}{F}{}^{ab}=x^{ak}y^{b0}-x^{a\ k-1}y^{b1}+\cdots\cdots+(-1)^k x^{a0}y^{bk}. \\ k=min.\ (k(a),\ k(b)),\quad a,\ b=1,\ 2,\ \ldots,\ m. \end{cases}$$

$$(4) \qquad \begin{cases} x^S z_T \quad S=01,\ \ldots,\ 0n,\ 10,\ \ldots,\ m0 \quad T=01,\ \ldots,\ 0n,\ 1k(1),\ \ldots,\ mk(m) \\ \overset{1}{F}{}^a_b=x^{a1}z_{bk(b)}+x^{a0}z_{b\ k(b)-1}, \\ \overset{2}{F}{}^a_b=x^{a2}z_{bk(b)}+x^{a1}z_{b\ k(b)-1}+x^{a0}z_{b\ k(b)-2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \overset{k}{F}{}^a_b=x^{ak}z_{bk(b)}+\cdots\cdots+x^{a0}z_{b\ k(b)-k}, \\ k=Min.\ (k(a),\ k(b)) \quad a,\ b=1,\ 2,\ \ldots,\ m. \end{cases}$$

are easily seen to be the bases of $n(Z)$, $n(Z_{0,2})$ and $n(Z_{1,1})$ respectively.

Now from the condition that the invariants (2) and (3), or (2) and (4), are also invariants of Z', we can easily show that Z' is a polynomial of Z without constant term. Q.E.D.

We note here the fact that $(Z_{0,2})_{0,2}=Z_{0,4}$ etc. Then since $Z_{m,n}$ is obviously nilpotent together with Z, we get immediately the following Theorem in virtue of Lemma 1′ and Lemma 2.

THEOREM 1′. *If Z is a nilpotent matrix and Z' is a matrix satisfying*

$$n(Z')\supset n(Z),\ n(Z'_{0,2})\supset n(Z_{0,2}),\ n(Z'_{0,4})\supset n(Z_{0,4});$$

or $n(Z')\supset n(Z)$, $n(Z'_{1,1})\supset n(Z_{1,1})$, $n(Z'_{2,2})\supset n(Z_{2,2})$; etc., then $Z'=q(Z)$, where $q(x)$ is a polynomial of the form (1). (Accordingly Z' is a replica of Z.)

§ 2. In his paper of linear representability of Lie algebras, I. Ado proved the following theorem[1].

THEOREM. *Let $\mathfrak{L}$ be a linear Lie algebra over a field of characteristic 0 and let $\mathfrak{J}$ be an ideal of $\mathfrak{L}$ composed only of nilpotent matrices. Then the factor algebra $\mathfrak{L}/\mathfrak{J}$ is also linear representable.*

Here we shall give an algebraic proof of a somewhat extended theorem. First let $\mathfrak{L}$ be a linear Lie algebra and let $X_1, \ldots, X_r$ be a basis of $\mathfrak{L}$. Putting

$$n(\mathfrak{L}_{m,n})=\bigcap_{i=1}^{r} n((X_i)_{m,n}),$$

we see that $n(\mathfrak{L}_{m,n})$ is independent of the choice of the basis of $\mathfrak{L}$. Now we shall prove the following

THEOREM 2. *Let $\mathfrak{L}$ be a linear Lie algebra composed of nilpotent matrices of degree k with a basis $X_1, \ldots, X_r$, and M a sufficiently large positive integer, e.g. $M=4^r$. Let now U be a matrix of degree k such that*

1) I. Ado, "On the representations of finite continuous group by means of linear transformations," (in Russian) Izvestia Kazan, 7 (1934-35).

$$n(U_{0,m})\supset n(\mathfrak{L}_{0,m}),\quad m=1,\ 2,\ \dots,\ M.$$

Then $U=\sum_{i=1}^{r} X_i'$, *where* X_i' *is a replica of* X_i. *Hence in particular if the field* P *is of characteristic* 0, *we have* $U\in\mathfrak{L}$. *In other words, a linear Lie algebra composed of nilpotent matrices over a field of characteristic* 0 *is defined by its invariants.*

PROOF (by induction with respect to r). When $r=1$, the theorem reduces to Theorem 1′ itself. Assume that for $\mathfrak{L}$ of dimension $r-1$ the theorem has already been established. The correspondence defined by

$$X\to\bar{X}=\begin{Vmatrix} X & & & & \\ & X_{0,2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & X_{0,h}\end{Vmatrix}$$

gives a representation of the Lie algebra composed of all matrices of degree k, and in particular that of $\mathfrak{L}$, in the space $\mathfrak{T}=\sum_{t=1}^{h}\mathfrak{T}_{0,t}$. Here we shall take h as 4^{r-1}. Since the Lie algebra $\mathfrak{L}$ is solvable, we can suppose that $\mathfrak{L}^*=PX_1+\dots+PX_{r-1}$ is an ideal of $\mathfrak{L}=PX_1+\dots+PX_r$. Then the space

$$\mathfrak{N}=\sum_{t=1}^{k} n(\mathfrak{L}^*_{0,t})$$

is allowable for $\mathfrak{L}$ in this representation, and X_r induces a nilpotent matrix $\tilde{X}_r$ on $\mathfrak{N}$.

Now from

$$n((\tilde{X}_r)_{0,t})\subset\sum_{m=1}^{M} n(\mathfrak{L}_{0,m})\quad t=1,\ 2,\ 4,\quad M=4^r,\ \text{follows}$$

$$n((\tilde{X}_r)_{0,1})\subset n(\tilde{U}_{0,t}),\qquad t=1,\ 2,\ 4.$$

From this and the nilpotency of $\tilde{X}_r$, as in the proof of Theorem 1, we can easily prove that $\mathfrak{N}$ is allowable for $\bar{U}$, and that the induced matrix $\tilde{U}$ is a replica of $\tilde{X}_r$. Hence $\tilde{U}=q(\tilde{X}_r)$, where $q(x)$ is of the form (1). Now putting $U-q(X_r)=V$, we have $\bar{V}\mathfrak{N}=0$, i.e.

$$n(V_{0,m})\supset n(\mathfrak{L}^*_{0,m}),\quad m=1,\ 2,\ \dots,\ 4^{r-1}.$$

Now by the assumption of induction we obtain that

$$V=\sum_{i=1}^{r-1} X_i',$$

where X_i' is a replica of X_i. Hence we get $U=\sum_{i=1}^{r} X_i'$. Q.E.D.

From this theorem follows Ado's immediately. Because, if the ideal $\mathfrak{J}$ is of dimension r, then the representation of $\mathfrak{L}$ induced on the allowable space $\sum_{m=1}^{M} n(\mathfrak{J}_{0,m})$ for $\mathfrak{L}$, is a faithful representation of $\mathfrak{L}/\mathfrak{J}$ for M not less than 4^r.