On the Inequality of Ingham and Jessen.

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Minkowski's inequality was formulated by Ingham and Jessen in the following symmetrical form.[1]

Let A be a m-rowed and n-columned matrix with non-negative elements

$$A=\begin{pmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \vdots & \vdots \qquad\quad \vdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{pmatrix}$$

Then the following inequality holds:

$$\Big(\sum_{\mu=1}^{m}\Big(\sum_{\nu=1}^{n} a_{\mu\nu}^{r}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}} \leqq \Big(\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{m} a_{\mu\nu}^{s}\Big)^{\frac{r}{s}}\Big)^{\frac{1}{r}},$$

if $0<r<s<\infty$.

We write this inequality in the form of quotient

$$1 \leqq \frac{\Big(\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{m} a_{\mu\nu}^{s}\Big)^{\frac{r}{s}}\Big)^{\frac{1}{r}}}{\Big(\sum_{\mu=1}^{m}\Big(\sum_{\nu=1}^{n} a_{\mu\nu}^{r}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}}},$$

and wish to evaluate this quotient form on the right-hand side. The result is the

Theorem

$$\frac{\Big(\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{m} a_{\mu\nu}^{s}\Big)^{\frac{r}{s}}\Big)^{\frac{1}{r}}}{\Big(\sum_{\mu=1}^{m}\Big(\sum_{\nu=1}^{n} a_{\mu\nu}^{r}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}}} \leqq \mathrm{Min.}\,(m, n)^{\frac{1}{r}-\frac{1}{s}}.$$

The constant on the right side is the best possible.

At first we suppose $r=1$, $s>1$ and prove a simple lemma, which can be easily verified with the help of elementary calculus.

Lemma. Let $0\leqq x\leqq c$, $0\leqq y\leqq c$ be variables, whose sum is constant: $x+y=c$, then the function

$$f(x, y)=(x^s+a^s)^{\frac{1}{s}}+(y^s+b^s)^{\frac{1}{s}}$$

attains its maximum only at the extremity of the interval $(0, c)$.

Proof.

$$f(x, y)=f(x, c-x)=g(x)$$

1) Hardy-Littlewood-Polya, Inequalities, p. 31

$$g'(x)=\{x(x^s+a^s)^{-\frac{1}{s}}\}^{s-1}-\{y(y^s+b^s)^{-\frac{1}{s}}\}^{s-1}$$

Hence $g'(0)<0$, $g'(c)>0$ and $g'(x)=0$, when $x:a=y:b$.

Therefore we obtain the lemma. q.e.d.

Because the numerator and donominator of the quotient are homogeneous functions of 1st order, so it suffices to study the behaviour of numerator under the condition $(\sum_{\mu=1}^{m}(\sum_{\nu=1}^{n}a_{\mu\nu}^{r})^{\frac{s}{r}})^{\frac{1}{s}}=1$. As the set of A for which $(\sum_{\mu=1}^{m}(\sum_{\nu=1}^{n}a_{\mu\nu}^{r})^{\frac{s}{r}})^{\frac{1}{s}}=1$, is evidently compact, so the continuous function $(\sum_{\nu=1}^{n}(\sum_{\mu=1}^{m}a_{\mu\nu}^{s})^{\frac{r}{s}})^{\frac{1}{s}}$ attains its maximum.

We suppose that for some i, j, k, l, $(i\neq j,\ k\neq l)$ 3 elements a_{ik}, a_{il}, $a_{j,l}$ are not 0. Then we will show that for such A the quotient does not attain the maximum.

Without loss of generality we can put $i=1$, $k=1$, $l=2$, $j=2$, because our problem remains invariant under permutations of rows and columns. Put

$$\varphi(A)=\frac{(a_{11}^s+a_2^s+\ldots\ldots)^{\frac{1}{s}}+(a_{12}^s+a_{22}^s+\ldots\ldots)^{\frac{1}{s}}+\ldots\ldots}{\{(a_{11}+a_{12}+\ldots\ldots)^s+(a_{21}+a_{22}+\ldots\ldots)^s+\ldots\ldots\}^{\frac{1}{s}}}$$

And in $\varphi(A)$ put $a_{11}=x$, $a_{12}=y$, $(a_{21}^s+\ldots\ldots+a_{m1}^s)^{\frac{1}{s}}=a$, $(a_{22}^s+\ldots\ldots+a_{m2}^s)^{\frac{1}{s}}=b>0$, and apply the above lemma, then $\varphi(A)$ is not maximum. Therefore the maximum value of $\varphi(A)$ is attained only when such "hook-shaped" system of 3 non-zero elements does not appear in A. After suitable permutations of rows and columns A can be brought to the following form: for example

$$\begin{pmatrix} * & * & 0 & 0 & \vdots \\ 0 & 0 & * & 0 & \vdots \\ 0 & 0 & * & 0 & \vdots \\ 0 & 0 & 0 & * & \vdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

Stars denote the non-zero elements.

Grouping the non-zero elements in the same row or column, we obtain k blocks of non-zero elements. When the block $(a_{i1}, a_{i2},\ldots a_{ij})$ is situated in some row, put

$$a_{i1}+a_{i2}+\ldots+a_{ij}=x_1.$$

When the block $(a_{lq}, a_{l,q+1}\ldots a_{lp})$ is situated in some column, put $(a_{lq}^s+\ldots+a_{lp}^s)^{\frac{1}{2}}=x_2$.

After such transformation our quotient is brought to the following form

$$\varphi(A)=\frac{x_1+x_2+\dots+x_k}{(x_1^s+x_2^s+\dots+x_k^s)^{\frac{1}{s}}}.$$

$\varphi(A)$ attains maximum when and only when

$$x_1=x_2=\dots=x_k.$$

For such values $\varphi(A)=k^{1-\frac{1}{s}}$.

Evidently $k\leqq \text{Min.}(m, n)$, so we obtain

$$\varphi(A)\leqq \text{Min.}(m, n)^{1-\frac{1}{s}}.$$

If in the above reasoning we put $a_{\mu\nu}^r$ instead of $a_{\mu\nu}$ and s/r instead of s, then we obtain easily

$$\varphi(A)\leqq \text{Min.}(m, n)^{\frac{1}{r}-\frac{1}{s}}$$

This constant is actually attained by

$$A=\begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix}$$

q.e.d.

Our theorem is easily extended to the case where one of m and n is infinity, and to the case where one of them is a continuous variable. But such an extension is almost trivial.

An inequality due to Pólya and Szegö[2] seems to lie in the same direction of idea.

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2) Pólya-Szegö, Aufgaben und Lehrsätze, 1, p. 57. Aufgabe 92.