22. On the Behaviour of the Boundary of Riemann Surfaces. I.

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We shall define the generalized harmonic measure of the boundary of a given Riemann surface and then classify it into two types.

Theorem 1. (An extension of R. Nevanlinna's theorem[1]*) Let F be a Riemann surface with a finite number of sheets spread over the z-plane and have Green's function and E be the set of its all accesible boundary points. We map the universal covering surface F^∞ of F on $|w|<1$, then the measure of e on $|w|=1$, which corresponds to E, is 2π.*

Proof. The mapping function $z=f(w)$ is automorphic with respect to a Fuchsian group G and let F be mapped on a fundamental domain D_0, which contains $w_0=0$. Let w_n be its equivalent. Since F has Green's function, we have by Poincaré's theorem[2]

$$\sum_{n=0}^{\infty}(1-|w_n|)<\infty\,. \tag{1}$$

We shall show that characteristic function $T(r)$ of $f(w)$ is bounded. Let a_0 be any point in D_0 and a_n be its equivalent, then we have

$$\left|\frac{w_0-a_0}{1-\bar{a}_0\cdot w_0}\right|=\left|\frac{w_n-a_n}{1-\bar{a}_n\cdot w_n}\right|.$$

Hence

$$|a_0|=\left|\frac{0-a_0}{1-\bar{a}_0\cdot 0}\right|=\left|\frac{w_n-a_n}{1-\bar{a}_n\cdot w_n}\right|.$$

From this we can easily deduce that

$$\frac{1-|a_0|}{1+|a_0|}(1-|w_n|)\leqq 1-|a_n|\leqq\frac{1+|a_0|}{1-|a_0|}(1-|w_n|)\,. \tag{2}$$

With respect to a general meromorphic function $f(w)$, $N(r,a)$ and $\sum_{r_\nu\leqq r}(1-r_\nu(a))$, where $r_\nu(a)$ is the absolute value of a-point of $f(w)$, are reciprocally uniformly bounded for some set of a.

Now we apply it on our automorphic function $f(w)$. Denote by $n(z)$ the number of sheets of F above z. Since F consits of a finite number of sheets, the maximum of $n(z)$ when z varies on the

1) R. Nevanlinna: Eindeutige Analytische Funktionen. Berlin, (1936), p. 204.

2) H. Poincaré: Sur l'uniformisation des fonctions analytiques. Acta. Math, 31 (1907).

z-plane is finite k and let $n(a)=k$. Then a small disc K about a is covered k-times by F. Hence the part of F above K contains k discs ; $F_1, F_2, \dots, F_k$ consisting of only inner points of F, where a piece of a Riemann surface of $(z-z_0)^{\frac{1}{n}}$ above K is considered as n discs, and their transforms into D_0 are Jordan closed domains : $G_1, G_2, \dots, G_k$. Let d be the minimum distance between these k domains and $|w|=1$. The b-points of $f(w)$ for any point b in K are both in D_0 $\beta_1, \beta_2, \dots, \beta_k$, where $\beta_i \in G_i$ and outside D_0 all β_i^ν, where β_i^ν is equivalent point of β_i $(i=1, 2, \dots, k)$.

Now for $\beta_i (i=1, 2, \dots, k)$ we have by (2)

$$1-|\beta_i^\nu| \leqq \frac{2}{d}(1-|w_\nu|) ,$$

so that

$$\sum_{i=1}^{k}(1-|\beta_i^\nu|) \leqq \frac{2k}{d}(1-|w_\nu|).$$

Finally we have by (1)

$$\sum_{\nu=0}^{\infty}\sum_{i=1}^{k}(1-|\beta_i^\nu|) \leqq \frac{2k}{d}\sum_{\nu=0}^{\infty}(1-|w_\nu|) < \infty .$$

This left side is the summation of all b-points of $f(w)$ and it is obviously uniformly bounded irrespective of b in K. Hence as above stated $N(r, b)$ is too uniformly bounded in K.

On the other hand by R. Nevanlinna's theorem[3)] we have

$$T(r)=\int_K N(r, b)d\mu + 0\,(1) .$$

Hence $T(r)$ is bounded, q.e.d. Next since $T(r)$ is bounded, by R. Nevanlinna's theorem[4)] limit $f(w)$ exists almost everywhere on $|w|=1$, when w tends to $|w|=1$ nontangentially. Obviously this limiting values belong to E, so that measure of e on $|w|=1$ is 2π, q.e.d.

Next we shall, after R. Nevanlinna[5)], measure the boundary of a Riemann surface spread over the z-plane as follows.

In the first place we consider a connected piece $\overline{F}$ of F, which is bounded by a finite number of closed Jordan curves (Γ^i) consisting of only ordinary points of F. We call hereafter (Γ^i) the relative boundary of $\overline{F}$ with respect to F and the other boundary E of $\overline{F}$ which is the boundary of F, "proper". Now we approximate $\overline{F}$ as well-known by a sequence of Riemann surfaces : $\overline{F}_0 \subset \overline{F}_1 \subset \overline{F}_2 \subset \dots \subset \overline{F}_n \to \overline{F}$ such that $\overline{F}_n$ consists of a finite number of

3) R. Nevanlinna : loc. cit. 1), p. 171.

4) R. Nevanlinna : loc. cit. 1), p. 197.

5) R. Nevanlinna : loc. cit. 1), pp. 106-114, and Über die Lösbarkeit des Dirichletschen Problems für eine Rie: annsche Fläche. Göttingen Nachr. (1939)

sheets and is bounded by (Γ^i) and a finite number of closed Jordan curves (C_n^j), where C_n^j does not split up $\bar{F}$ into two pieces F_n', F_n'' abutting along C_n^j such that $\bar{F}_n < F_n'$ and F_n'' consists of inner points of $\bar{F}$. Next we consider a harmonic function $\bar{u}_n(z)$ on $\bar{F}_n$ with the next boundary condition ;

$$\bar{u}_n(z)=0 \quad \text{on } (\Gamma^i)\,, \quad \bar{u}_n(z)=1 \quad \text{on } (C_n^j)\,.$$

In fact we can find this function as follows. Since $\bar{F}_n$ is bounded by only closed Jordan curves, obviously $\bar{F}_n$ has Green's function and consists of a finite number of sheets. When we map the universal covering surface $\bar{F}_n^\infty$ of $\bar{F}_n$ on $|w|<1$, (Γ^i) and (C_n^i) correspond to arcs e and e_n respectively and by theorem 1 $me+me_n=2\pi$.

Let

$$U(w)=U(re^{i\theta})=\frac{1}{2\pi}\int_{e_n}\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}d\varphi\,.$$

Then $U(w)=1$ on e_n and $U(w)=0$ on e. We put $\bar{u}_n(z)=U(w)$, then $\bar{u}_n(z)$ is the required function. Obviously by the maximum principle

$$0<\bar{u}_{n+1}(z)<\bar{u}_n(z) \qquad \text{on } \bar{F}_n\,.$$

If $\bar{F}_n\to\bar{F}$, then by Harnack's theorem, $\lim_{n\to\infty}\bar{u}_n(z)=\bar{u}(z)$ is uniformly convergent on $\bar{F}$, so that $\bar{u}(z)$ is a bounded harmonic function on $\bar{F}$. We call $\bar{u}(z)$ "harmonic measuring function" belonging to $\bar{F}$ and $\bar{u}_n(z)$ its approximating function.

Definition. According to $\bar{u}(z)\equiv 0$ or $\not\equiv 0$, we call respectively after R. Neyanlinna[6] that the absolute harmonic measure of the proper boundary E of $\bar{F}$ is zero and $\bar{F}$ is "of the first kind", or that the absolute harmonic measure of E is positive and $\bar{F}$ is "of the second kind".

Considering the above stated process, it is easily seen that the nature of $\bar{u}(z)\equiv 0$ or $\not\equiv 0$ is not only independent of the selection of (C_n^j), but of a suitable slight deformation of (Γ^i).

When specially (Γ^i) consists of only one closed Jordan curve Γ, which separates F into two pieces F', F'' where F' consists of only ordinary points of F, we have next Myrberg-Tsuji's theorem.

Theorem 2. (*Morberg-Tsuji*)[7]

(*i*) *If F has no Green's function, then $\bar{u}(z)\equiv 0$.*

(*ii*) *If F has Green's function, ihen $\bar{u}(z)\not\equiv 0$.*

6) R. Nevanlinna: loc. cit. 5).

7) P. J. Myrberg: Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Flächen. Acta. Math. 61 (1938).

M. Tsuji: Some metrical theorems on Fuchsian groups. Japan. Journ. Math. 19 (1947), pp. 509-512.

Proof. Let $G_n(z, a)$ be a Green's function of F_n with a logarithmic singularity at a ordinary point a of F', where F_n is the connected piece of F bounded by (C_n^j) including a and Γ. Then

$$\bar{u}_n(z)=0 \quad \text{on } \Gamma, \quad \bar{u}_n(z)=1 \quad \text{on } (C_n^j), \quad G_n(z, a)=0 \quad \text{on } (C_n^j).$$

Then by the maximum principle

$$(1-\bar{u}_n(z)M_n \geqq G_n(z, a) \quad \text{on } \bar{F}_n, \text{ where } \quad M_n=\max_{\text{on } \Gamma}. G_n(z, a)>0.$$

Hence

$$(1-\bar{u}(z))M_n \geqq G_n(z, a) \quad \text{on } \bar{F}_n. \tag{3}$$

Next suppose that F_0 includes Γ. Since $G_n(z, a)-G_0(z, a)$ is obviously harmonic at a, it is harmonic on F_0. Hence by the maximum principle

$$\max_{\text{on } (C_n^j)}. (G_n(z, a)-G_0(z, a)) \geqq \max_{\text{on } \Gamma}. (G_n(z, a)-G_0(z, a)).$$

Namely

$$\max_{\text{on } (C_n^j)}. G_n(z, a) \geqq M_n-k, \quad \text{where} \quad k=\max_{\text{on } \Gamma}. G_0(z, a)>0.$$

Then by (3)

$$\left(1-\min_{\text{on } (C_n^j)}. \bar{u}(z)\right)M_n \geqq \max_{\text{on } (C_n^j)}. G_n(z, a) \geqq M_n-k.$$

Namely

$$\min_{\text{on } (C_n^j)}. \bar{u}(z) \leqq \frac{k}{M_n}.$$

Since by the hypothesis $G_n(z, a)\to\infty$ and $M_n\to\infty$ $(n\to\infty)$, then

$$\min_{\text{on } (C_n^j)}. \bar{u}(z)=0.$$

Namely $\bar{u}(z)\equiv 0$, q.e.d.

(ii) Let $G(z, a)$ be Green's function of F and $m=\min_{\text{on } \Gamma}. G(z, a)>0.$ Then since $\bar{u}_n(z)=0$ on Γ, $\bar{u}_n(z)=1$ on (C_n^j), by the maximum principle

$$\frac{m-G(z, a)}{m} \leqq \bar{u}_n(z) \qquad \text{on } \bar{F}_n.$$

Hence

$$\frac{m-G(z, a)}{m} \leqq u(z) \qquad \text{on } \bar{F}.$$

By the property of Green's function for $m>0$ we have $z_0 \in \bar{F}$ such that $G(z_0, a)<m$. For this z_0

$$\bar{u}(z_0)>0.$$

Hence $u(z)\not\equiv 0$, q.e.d.

Moreover we separate $\bar{F}$ into k connected pieces: $\bar{F}^1, \bar{F}^2, \ldots,$

$\overline{F}^k$, each of which has the relative boundary (Γ_m^i) and the propre boundary E_m, and let $\bar{u}^m(z)$ be measuring function of $\overline{F}^m$. Then we shall prove ;

Theorem 3. (*i*) *If* $\bar{u}(z)\equiv 0$, *then every* $\bar{u}^m(z)\equiv 0$.

(*ii*) *If every* $\bar{u}^m(z)\equiv 0$, *then* $\bar{u}(z)\equiv 0$.

Proof. (i) By the boundary condition :

$\bar{u}_n^m(z)<\bar{u}(z)$ on $\overline{F}_n^m$, where $\overline{F}_n^m$ is the connected piece of $\overline{F}^m$, which is enclosed by (Γ_m^i) and $(C_n^{j(m)})$. For $n\to\infty$

$$0\leqq\bar{u}^m(z)\leqq\bar{u}(z) \quad \text{on } \overline{F}^m, \ (m=1, 2, \dots, k).$$

Since by the hypothesis $u(z)\equiv 0$, then

$$\bar{u}^m(z)\equiv 0 \quad \text{on } \overline{F}^m, \ (m=1, 2, \dots, k), \text{ q.e.d.}$$

(ii) We suppose that $\bar{u}(z)\not\equiv 0$ and we put

$$M=\max_{\text{on all } (\Gamma_m^i)} \bar{u}(z)>0 .$$

Then

$$\bar{u}(z)-M\leqq\bar{u}_n^m(z) \quad \text{on } \overline{F}_n^m, \ (m=1, 2, \dots, k).$$

For $n\to\infty$

$$\bar{u}(z)-M\leqq\bar{u}^m(z) \quad \text{on } \overline{F}^m, \ (m=1, 2, \dots, k).$$

Since by the hypothesis every $\bar{u}^m(z)\equiv 0$, then

$$\bar{u}(z)\leqq M \quad \text{on every } \overline{F}^m. \tag{4}$$

On the other hand $\bar{u}(z)$ is harmonic on $\overline{F}_n$ and $\bar{u}(z)=0$ on (Γ^i) and $\bar{u}(z)>0$ on $(C_n^{j(m)})$, hence by the maximum principle we can find such a point z_0 on $(C_n^{j(m)})$, that

$$\bar{u}(z_0)>M.$$

This contradicts to (4) and therefore $\bar{u}(z)$ must be $\equiv 0$, q.e.d.

Finally we shall prove;

Theorem 4. Let F and $\overline{F}$ be respectively a Riemann surface spread over the z-plane and the w-plane, and both correspond in a one-one conformal manner by $w=f(z)$, $z=\varphi(w)$. If F is of the first kind or the second kind, its transform $\overline{F}$ must be respectively of the first kind or of the second kind.

Proof. We consider harmonic measuring function $\overline{U}(w)$ of $\overline{F}$ with its approximating function, $\overline{U}_n(w)$, then $\bar{u}_n(z)=\overline{U}_n(f(z))$ is surely a approximationg function, for $\overline{F}_n$ rnd $\overline{F}_n$ both consist of only inner points.

Hence for $n\to\infty\,\bar{u}_n(z)\to u(z)\equiv 0$, or $u(z)\not\equiv 0$, so that respectively $\overline{U}(w)\equiv 0$ or $U(w)\not\equiv 0$, q.e.d.