27. On the Radiation Pressure in a Planetary Nebula. I.

By Wasaburo UNNO.

(Comm. by Y. HAGIHARA, M.J.A., June 12, 1950.)

Abstract.

The radiation pressure of the Lyman α line-radiation in a planetary nebula is discussed. Zanstra's[1] idea of redistribution in frequency in the line-contour is taken into account in detail. The equation of transfer of the Lyman α radiation with redistribution mechanism is solved in contrast with Zanstra's rough treatment in which a definite form of emission and complete redistribution are assumed. The result obtained is found to be nearly the same as in Zanstra's theory. The radiation pressure due to the Lyman α radiation is so much reduced that we should be able to get rid of the blowing-up difficulty of planetary nebula in Ambarzumian's[2] theory. Thus it is confirmed that the complete redistribution is a good approximation to the solution of this problem.

1. The fundamental equation.

The basic equation for the transfer of the Lyman α radiation in the shell of a planetary nebula is taken to be

$$\cos\theta \frac{d I(\nu, z, \theta)}{\rho\, dz} = - I(\nu, z, \theta) \int \sigma\left(\nu\left[1 - \frac{1}{c}(vr)\right]\right) \psi(v)\, dv$$
$$+ \frac{1}{4\pi}\iint dv\, dr'\, \sigma\left(\nu\left[1 - \frac{1}{c}(vr)\right]\right) \psi(v)\, I\left(\nu\left[1 - \frac{1}{c}(vr)\right.\right.$$
$$\left.\left. + \frac{1}{c}(vr')\right], z, \theta'\right) + Se^{-\tau}\int \sigma\left(\nu\left[1 - \frac{1}{c}(vr)\right]\right) \psi(v)\, dv, \quad (1)$$

where z-axis is taken in the direction of direct radiation from the central star, $I(\nu, z, \theta)$ the intensity of the L_α radiation at an angle θ with the z-axis at the distance z from the inner boundary of the nebula, $\psi(v)\, dv$ is the well-known Maxwellian velocity distribution of the hydrogen atom in its ground state, σ the natural damping contour, and $Se^{-\tau}$ is assumed to be the amount of the L_α emission followed by the absorption of the Lyman continuum at the optical thickness τ:

$$S = \frac{\nu_c}{\nu} \frac{\kappa_c}{2\sigma_0} \frac{1}{4} S_c (1 - p),$$

where S_c is the intensity of the Lyman continuum at the inner boundary ν_c the frequency of Lyman limit, κ_c and σ_0 are the absorption coefficients at the Lyman limit and the line center of Doppler contour of the L_α. The first term of the right hand side in the equation (1) is due to the absorption and the second and the third

1) H. Zanstra: B. A. N., **11**, No. 401, 1, 1949.

2) V. A. Ambarzumian: M. N., **93**, 50, 1932. Y. Hagihara: Jap. J. Astr. Geophys., **15**, 1, 1938; **20**, 113, 1943. Hagihara & Hatanaka: *ibid.*, **19**, 135, 1942.

terms are due to the emission of frequency ν followed by the absorption of the diffuse L_α radiation and the Lyman continuum. The square bracketed factor preceded by ν is the effect of the Doppler shift due to the thermal motion of the atom. The integral factors in the first and the third terms give the Doppler contour. If we neglect the width of the natural line-contour we can simplify it in the following form:

$$\int \sigma\left(\nu\left[1-\frac{1}{c}(\boldsymbol{v}\boldsymbol{r})\right]\right)\psi(v)\,dv = \int \sigma\left(\nu\left[1-\frac{v_\zeta}{c}\right]\right)e^{-\frac{1}{2}mv_\zeta^2/kT}$$
$$\times\frac{1}{\sqrt{\pi}}\,d\left(\sqrt{\frac{m}{2kT}}\,v_\zeta\right) = \frac{1}{\sqrt{\pi}}\,e^{-\frac{1}{2}mc^2\frac{(\nu-\nu_0)^2}{\nu_0^2 kT}}\cdot\sigma(\nu_0)\cdot N_1$$
$$= \sigma_0\,e^{-\left(\frac{\Delta\nu}{D}\right)^2} = \sigma_0\,e^{-x^2} = \sigma_D(\nu)\,, \tag{2}$$

where ζ is the axis taken in the direction of radiation concerned and D is the Doppler width of the line. The second term is rather complex owing to the double integrals with respect to $\boldsymbol{v}$ and $\boldsymbol{r}'$. $\boldsymbol{r}$ and $\boldsymbol{r}'$ represent the directions of radiation concerned and the absorbed diffuse Lyman α radiation. We take the ζ-axis in the direction of $\boldsymbol{r}$ as before and use the cylindrical coordinates:

$\boldsymbol{v}(v,\varphi,v_\zeta)\,,\quad \boldsymbol{r}(\sin\vartheta',\phi',\cos\vartheta')\,,\quad$ then

$$\boldsymbol{v}\boldsymbol{r} = v_\zeta\,,$$
$$\boldsymbol{v}\boldsymbol{r}' = v_\zeta\cos\vartheta' + v\sin\vartheta'\cos(\phi'-\varphi) = v_\zeta\cos\vartheta' + v\sin\vartheta'\cos\phi.$$

We take the natural line-contour much sharper than the Doppler so that

$$\sigma\left(\nu\left[1-\frac{v_\zeta}{c}\right]\right) = \sigma(\nu_0) \qquad \text{when} \qquad \nu\left[1-\frac{v_\zeta}{c}\right] = \nu_0\,,$$

and $\qquad\qquad = 0 \qquad$ otherwise.

Then we can integrate with respect to v_ζ and φ, and the term will easily reduce to the form:

$$\frac{1}{4\pi}\,\sigma_D(\nu)\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} I(\nu_0+\mu\Delta\nu+2\beta D\sqrt{1-\mu^2}\cos\phi, z, \bar{\theta}')\,2\beta e^{-\beta^2}\,d\beta\,d\mu\,d\phi,$$

where $2D = \nu_0\dfrac{v_0}{c}$ and $mv_0^2 = kT$. $\bar{\theta}'$ is the complicated function of angular variables, but we neglect the variation of I with $\bar{\theta}'$ because the integrations are carried out with these angular variables. Therefore we can replace I by J, the source function, and we get

$$\cos\theta\frac{dI(\nu,z,\theta)}{\rho\,dz} = \sigma_D(\nu)\Big[-I(\nu,z,\theta) + Se^{-\tau}$$
$$+\frac{1}{4\pi}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} J(\nu_0+\mu\Delta\nu+2\beta D\sqrt{1-\mu^2}\cos\phi, z)\,2\beta e^{-\beta^2}\,d\beta\,d\mu\,d\phi\Big]. \tag{3}$$

Next we introduce new quantities s, $p(x)$, and u by the equations,

$$ds = \rho\,\sigma_0\,dz\,, \qquad \sigma_D(\nu) = \sigma_0\,e^{-x^2} = \sigma_0\,p(x)$$
$$d\tau = u\,ds \quad \text{or} \quad \tau = us\,.$$

u is the ratio of κ_c and σ_0 and is of the order of 10^{-4}. Writing equation (3) with the new variables, we get

$$\cos\theta\,\frac{dI(x,s,\theta)}{ds} = p(x)\Big[-I(x,s,\theta) + Se^{-us} + \frac{1}{4\pi}\int_0^{\infty}\int_{-1}^{+1}\int_0^{2\pi} J(x\mu+2\beta\sqrt{1-\mu^2}\cos\phi, s)\,2\beta e^{-\beta^2}\,d\theta\,d\mu\,d\phi\Big] \qquad (3')$$

Making use of the Eddington's approximation, we obtain the equations which determine the flux F and the source function J,

$$F(x,s) = -\frac{4}{3}\,\frac{1}{p(x)}\,\frac{dJ(x,s)}{ds}\,, \qquad (4)$$

and

$$\frac{d^2J(x,s)}{ds^2} = 3p(x)^2\Big[J(x,s) - Se^{-us} - \frac{1}{4\pi}\int_0^{\infty}\int_{-1}^{+1}\int_0^{2\pi} J(x\mu+2\beta\sqrt{1-\mu^2}\cos\phi, s)\,2\beta e^{-\beta^2}\,d\beta\,d\mu\,d\phi\Big] \qquad (5)$$

This is the fundamental equation to be solved.

2. The solution of the first approximation.

The line contour is symmetrical with respect to the frequency of the line center, therefore, we need only to treat one side of it. We divide the line contour into two frequency ranges; the nebula is opaque and semi-transparent for the radiations of the first and the second frequency ranges respectively. Then the equation (5) becomes the simultaneous equations, viz.,

$$\frac{d^2(I,s)}{ds^2} = q(I)^2\Big[(I,s) - \Big\{(1-\alpha_I)(I,s) + \alpha_I(II,s)\Big\} - Se^{-us}\Big],$$

$$\frac{d^2(II,s)}{ds^2} = q(II)^2\Big[(II,s) - \Big\{\frac{1}{2}[(1-\alpha'_{II})(II,s) + \alpha'_{II}(I,s)] + \frac{1}{2}[(1-\alpha''_{II})(I,s) + \alpha''_{II}(II,s)]\Big\} - Se^{-us}\Big]. \qquad (6)$$

where $(\ \ ,s)$ and $q(\ )^2$ stand for J and $3p^2$ in the respective frequency range, and $\alpha's$ are the redistribution factors. The redistribution factors represent the effect due to the tangential thermal motion term $2\beta\sqrt{1-\mu^2}\cos\phi$ in equation (5). Let the widths of the ranges I and II be $2w$, then $\alpha's$ in the above equations are roughly determined by the following equations:

$$\alpha_I = \frac{1}{2\pi}\int_0^\infty\int_0^{+1}\int_0^{2\pi} 2\beta e^{-\beta^2}\, d\beta\, d\mu\, d\phi$$

$$3w > 2\beta\sqrt{1-\mu^2}\cos\phi > w$$

$$\alpha'_{II} = \frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} 2\beta e^{-\beta^2}\, d\beta\, d\phi\,, \quad \mu = \frac{3}{4}\,,$$

$$-3w < 2\beta\sqrt{1-\mu^2}\cos\phi < -w$$

$$\alpha''_{II} = \frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} 2\beta e^{-\beta^2}\, d\beta\, d\phi\,, \quad \mu = \frac{1}{4}\,.$$

$$3w > 2\beta\sqrt{1-\mu^2}\cos\phi > w \tag{7}$$

Equations (6) now become

$$\frac{d^2(I,s)}{ds^2} = q(I)^2\left[\alpha_I\{(I,s)-(II,s)\} - Se^{-us}\right]$$

$$\frac{d^2(II,s)}{ds^2} = q(II)^2\left[-\alpha_{II}\{(I,s)-(II,s)\} - S^{-us}\right]. \tag{8}$$

where $\alpha_{II} = \frac{1}{2}(1+\alpha'_{II}-\alpha''_{II})$.

Equations (8) are the simultaneous linear differential equations, therefore the solutions are obtained easily.

$$\begin{aligned}\frac{(I,s)}{S} = {} & \frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)}{u}q(II)(s_1-s) - \frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)^2}{u^2}(e^{-us}-e^{-us_1}) \\ & + \frac{1}{\alpha}e^{-us} + \frac{2}{\sqrt{3}}\frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)}{u}(1-e^{-us_1}) - \frac{u}{\alpha_I\omega}e^{-\omega s} \\ & - \frac{1}{2\sqrt{\alpha_I}+\sqrt{3}}\left[\frac{2(\alpha_I+\alpha_{II})}{\alpha_I}\frac{q(II)}{u}(1-e^{-us_1}) + \frac{\sqrt{3}}{\alpha_I}e^{-us_1}\right]e^{-\omega(s_1-s)},\end{aligned} \tag{9a}$$

$$\begin{aligned}\frac{(II,s)}{S} = {} & \frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)}{u}q(II)(s_1-s) - \frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)^2}{u^2}(e^{-us}-e^{-us_1}) \\ & - \frac{2}{\sqrt{3}}\frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)}{u}(1-e^{-us_1}) - \frac{\alpha_{II}}{\alpha_I}\frac{uq(II)^2}{\omega^3}e^{-\omega s} \\ & + \frac{1}{2\sqrt{\alpha_I}+\sqrt{3}}\frac{\alpha_{II}}{\alpha_I}\frac{q(II)^2}{\omega^2}\Big[2(\alpha_I+\alpha_{II})\frac{q(II)}{u}(1-e^{-us_1}) \\ & + \sqrt{3}e^{-us_1}\Big]e^{-\omega(s_1-s)}\end{aligned} \tag{9b}$$

and

$$\begin{aligned}\frac{1}{S}\frac{1}{q(I)}\frac{d(I,s)}{ds} = {} & -\frac{\alpha_I+\alpha_{II}}{\alpha_I}\frac{q(II)^2}{uq(I)}(1-e^{-us}) - \frac{u}{\alpha_I q(I)}(e^{-us}-e^{-\omega s}) \\ & - \frac{1}{2\sqrt{\alpha_I}+\sqrt{3}}\frac{\omega}{\alpha_I q(I)}\Big[2(\alpha_I+\alpha_{II})\frac{q(II)}{u}(1-e^{-us_1}) \\ & + \sqrt{3}\,e^{-us_1}\Big]e^{-\omega(s_1-s)}\end{aligned} \tag{10a}$$

$$\frac{1}{S}\frac{1}{q(II)}\frac{d(II,s)}{ds} = -\frac{a_I+a_{II}}{a_I}\frac{q(II)}{u}(1-e^{-us}) + \frac{a_{II}}{a_I}\frac{uq(II)}{\omega^2}e^{-\omega s}$$
$$+\frac{1}{1\sqrt{a_I}+\sqrt{3}}\frac{a_{II}}{a_I}\frac{q(II)}{\omega}\left[2(a_I+a_{II})\frac{q(II)}{u}(1-e^{-us_1})\right.$$
$$\left.+\sqrt{3}\,e^{-us_1}\right]e^{-\omega(s_1-s)}\,, \qquad (10b)$$

where $\omega^2 = a_I q(I)^2$.

On obtaining the solutions (9), we have used Milne's boundary conditions of the planetary nebula:

$$F(0) = 0 \text{ or } \frac{d(\ ,s)}{ds} = 0 \text{ at } s = 0 \quad \text{for the inner boundary,}$$

and

$$F(s_1) = 2J(s_1) \text{ or } \frac{d(\ ,s)}{ds}\frac{\sqrt{3}}{2}q(\ )(\ ,s) = 0 \text{ at } s = s_1 \text{ for the}$$

outer boundary, and further we have omitted the negligible small terms by comparing, with each other, the order of magnitudes of the quantities, e.g.,

$$q(I) \sim 1\,,\qquad q(II) \sim 10^{-4}\,,\qquad \omega \sim 10^{-1}\,,$$
$$a's \sim 10^{-1}\,,\qquad u \sim 10^{-4}\,,\qquad s_1 \sim 10^{4}\,.$$

3. The second approximation and summary.

The second approximation is obtained by subdividing the contour into several regions and putting the solutions of the first approximation into the terms of the right hand side of equation (5) with the exception of the terms of the frequency range which we shall determine. And the solution is obtained easily. In spite of the more complex appearance of the solution, it is not much different in nature from the first approximation. It may be sufficient for obtaining a good result if we take the value of q in the solution of the first approximation appropriate for the frequency range considered, instead of the solution of the second approximation. Hence we omit the expression of the solution of the second approximation.

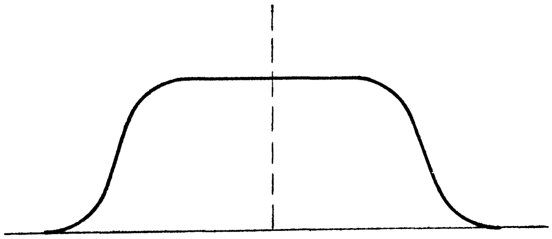

Fig. 1. The Source Function.

We are interested in the source function, the radiation pressure, and the radiation flux mainly in the central part of the nebula. For the outer part of the nebula more detailed discussion is required, which will be discussed in a subsequent paper. The source function is very flat as is seen from the expression (9) and Fig. 1. The flux of the line radiation which is proportional to

$$\frac{1}{q}\frac{d(\ ,s)}{ds}$$

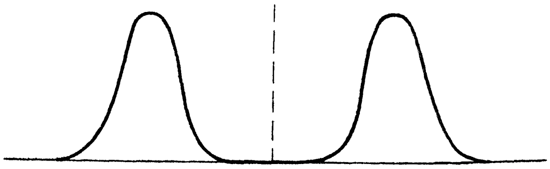

Fig. 2. The Flux.

is very little ($\sim 10^{-4}$) in the line center compared with the line wing where q is of the order of 10^{-4}, and is illustrated in Fig. 2. Hence the radiation pressure, being proportional to $\int_0^\infty \frac{d(\ ,s)}{ds} d\nu$, is 10^{-4} times smaller than that of the old theory of Ambarzumian,[2] and comparable that of Lyman continuum. This result is shown thus to be nearly the same as that of Zanstra's[1] theory in which the complete redistribution of frequency in the line is assumed. Therefore, it is found from our result that the complete redistribution approximation is fairly appropriate for treating the radiation pressure due to the line radiation.