56. *On the Zeros of Dirichlet's L-Functions.*

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We put $h=\varphi(k)$ where k is a positive integer and $\varphi(k)$ is Euler's function. Let $\chi(n)$ denote one of the h Dirichlet's characters with modulus k. $\bar{\chi}$ is the conjugate complex character of χ. $\zeta(s,w)$ and $L(s,\chi)$ denote the functions defined for $\sigma>1$ by $\sum_{n=0}^{\infty}(n+w)^{-s}$ and $\sum_{n=1}^{\infty}\chi(n)n^{-s}$ respectively, where $0<w\leqq 1$ and $s=\sigma+ti$. Throughout the paper, the notations $A\ll B$ and $A=O(B)$ for $B>0$ show that $|A|\leqq KB$, where K is a positive absolute constant.

We know from the recent work of *Rodosskiĭ* ([11], Theorem 1.) that the number of $L(s,\chi)$ which have a zero in the rectangle

$$1-\frac{\psi(k)}{\log kT}\leqq\sigma\leqq 1,\quad |t-T_1|\leqq K\log^2 kT$$

where $\frac{1}{4}\log k\geqq\psi(k)\geqq\log\log k$ and $T=|T_1|+2$ does not exceed $B\exp(A\,\psi(k)+5\log\log kT)$. From this we are able to deduce that the total number of zeros of all the L-functions with modulus k in the above rectangle does not exceed

$$C\exp(A\,\psi(k)+8\log\log kT) \tag{1}$$

where A, B, C and K are positive absolute constants.

The aim of this paper is to estimate the total number $N(\alpha,T)$ of zeros of all the L-functions with modulus k in the rectangle

$$\alpha\leqq\sigma\leqq 1,\quad |t|\leqq T$$

using Ingham's method [7]. The main result is that, if

$$\zeta(\tfrac{1}{2}+ti,w)-w^{-\frac{1}{2}-ti}=O(|t|^c) \tag{2}$$

where c is a positive absolute constant, then

$$N(\alpha,T)=O\{(k^4T^{4c}(T+k)^2)^{1-\alpha}\log^8 kT\}$$

for $\frac{1}{2}\leqq\alpha\leqq 1$, $T\geqq 2$. From this we are also able to deduce (1) and so *Rodosskiĭ's* main theorem ([11], Theorem 2.) in the theory of primes in an arithmetic progression.

We use some well known theorems in the theory of functions in the following forms.

Theorem A. (*Jensen*, [6], Theorem D, p. 49.) Suppose that

$H(s)$ is regular in the circle $|s-s_0|\leqq R$, and has ν zeros in $|s-s_0|\leqq r\ (<R)$. Then if $H(s_0)\neq 0$

$$\left(\frac{R}{r}\right)^{\nu}\leqq\frac{M}{|H(s_0)|}$$

where M is the maximum of $|H(s)|$ on $|s-s_0|=R$.

Theorem B. *Littlewood,* [13], p. 132.) Let C denote the rectangle bounded by the lines $\sigma=\sigma_1$, $\sigma=\sigma_2$ $(\sigma_1<\sigma_2)$, $t=T$, $t=-T$. Let $h(s)$ be analytic and not zero on C, and meromorphic inside it. Let $N^*(\alpha)$ denote the excess of the number of zeros of $h(s)$ over the number of poles in the part of the rectangle where $\sigma>\alpha$. Then

$$2\pi\int_{\sigma_1}^{\sigma_2}N^*(\sigma)d\sigma=\int_{-T}^{T}\{\log|h(\sigma_1+ti)|-\log|h(\sigma_2+ti)|\}dt$$
$$+\int_{\sigma_1}^{\sigma_2}\{\arg h(\sigma+Ti)-\arg h(\sigma-Ti)\}d\sigma.$$

Theorem C. (*Doetsch,* [3].) Suppose that $g(s,\chi)$ is regular and $G(s)=\sum_{\chi}|g(s,\chi)|^2$ is bounded in the strip $\sigma_1\leqq\sigma\leqq\sigma_2$. We write $M(\sigma)=\sup_{\Re s=\sigma}G(s)$ then

$$M(\sigma)\leqq M(\sigma_1)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}}M(\sigma_2)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}$$

Theorem D. (*Hardy-Ingham-Pólya,* [4].) Suppose that $g(s,\chi)$ is regular and $G(s)=\sum_{\chi}|g(s,\chi)|^2$ is bounded in the strip $\sigma_1\leqq\sigma\leqq\sigma_2$. We write $J(\sigma)=\int_{-\infty}^{\infty}G(\sigma+ti)\,dt$. If $J(\sigma_1)$ and $J(\sigma_2)$ are convergent, then $J(\sigma)$ is convergent and

$$J(\sigma)\leqq J(\sigma_1)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}}J(\sigma_2)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}$$

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1. On N(α, T).

Let $\mu(n)$ denote Möbius's function. We define $Q(s,\chi,z)=\sum_{n<z}\chi(n)\mu(n)n^{-s}$, $f(s,\chi,z)=L(s,\chi)Q(s,\chi,z)-1$, $h(s,\chi,z)=1-f^2(s,\chi,z)$, $K(\sigma,T,z)=\text{Max}\ (T-1.5\leqq|t|\leqq T+1.5)\sum_{\chi}|f(\sigma+ti,\chi,z)|^2$, and $I(\sigma,T,z)=\int_{-T}^{T}\sum_{\chi}|f(\sigma+ti,\chi,z)|^2\,dt$.

Lemma 1. If $\frac{1}{2}+2\delta\leqq\alpha<1$ $(\delta>0)$, then

$$N(\alpha,T)\ll\frac{1}{\delta}\Big\{I(\alpha-\delta,T,z)+I(2,T,z)\Big\}+$$
$$+\frac{1}{\delta^2}\Big\{\text{Max}(\alpha-2\delta\leqq\sigma\leqq4)\,K(\sigma,T,z)-\log(2-\exp K(2,T,z))\Big\}.$$

Proof. Let m be the number of zeros of the function

$$H(s, T, z) = \prod_{\chi} h(s+Ti, \chi, z) + \prod_{\chi} h(s-Ti, \chi, z)$$

on the segment $\alpha < \sigma < 2$, $t=0$. Hence m cannot exceed $\nu(\alpha, T, z)$ the number of zeros of $H(s, T, z)$ in the circle $|s-2| \leq 2-\alpha$. It is well known that

$$\left|\sum_{\chi} \arg h(\alpha+Ti, x, z)\right| \leq (m+1)\pi \ll \nu(\alpha, T, z). \tag{3}$$

On the other hand

$$\left(\frac{2-\alpha+\delta}{2-\alpha}\right)^{\nu(\alpha, T, z)} \leq \operatorname{Max}(|s-2| = 2-\alpha+\delta)\left|\frac{H(s, T, z)}{H(2, T, z)}\right| \tag{4}$$

by Theorem A. Further we have

$$\begin{aligned} H(2, T, z) &= 2\,\Re \prod_{\chi} h(2+Ti, \chi, z) = 2\,\Re \prod_{\chi}\{1-f^2(2+Ti, \chi, z)\} \\ &\geq 2\{1-(\prod_{\chi}(1+|f(2+Ti, \chi, z)|^2)-1)\} \\ &\geq 4-2\exp\sum_{\chi}|f(2+Ti, \chi, z)|^2, \end{aligned} \tag{5}$$

$$\begin{aligned} &\operatorname{Max}(|s-2| = 2-\alpha+\delta)\,|H(s, T, z)| \leq \exp\{\operatorname{Max}(|s-2| \leq 2 \\ &\quad -\alpha+\delta)\sum_{\chi}|f(s+Ti, \chi, z)|^2\} + \exp\{\operatorname{Max}(|s-2| \leq 2 \\ &\quad -\alpha+\delta)\sum_{\chi}|f(s-Ti, \chi, z)|^2\} \leq 2\exp\{\operatorname{Max}(\alpha-\delta \leq \sigma \leq 4 \\ &\quad -\alpha+\delta)\,K(\sigma, T, z)\}. \end{aligned} \tag{6}$$

From (4), (5) and (6) we get

$$\begin{aligned} \nu(\alpha, \pm T, z) \ll \frac{1}{\delta}\Big\{&\operatorname{Max}(\alpha-\delta \leq \sigma \leq 4-\alpha+\delta)\,K(\sigma, T, z) \\ &+\log 2-\log(4-2\exp K(2, T, z))\Big\} \end{aligned}$$

Now we put

$$\sigma_1 = \alpha-\delta,\ \sigma_2 = 2,\ h(s) = h(s, \chi, z),\ N^*(\alpha) = N^*(\alpha, T, \chi)$$

in Theorem B. We write $N^*(\alpha, T) = \sum_{\chi} N^*(\alpha, T, \chi)$. Noting that $N^*(\alpha, T) \leq \frac{1}{\delta}\int_{\alpha-\delta}^{\alpha} N^*(\sigma, T)\,d\sigma \leq \frac{1}{\delta}\int_{\alpha-\delta}^{2} N^*(\sigma, T)\,d\sigma$ and $N(\alpha, T) \leq N^*(\alpha, T)$ for $\frac{1}{2}+2\delta \leq \alpha < 1$, we obtain the result.

2. On $\rho(n, z)$.

We define

$$\rho(n) = \rho(n, z) = \begin{cases} \sum_{\substack{d|n \\ d \leq \frac{n}{z}}} 1. & (n, k) = 1 \\ 0 & (n, k) > 1 \end{cases} \qquad \text{for } z \geq k,$$

and

$$D(x) = D(x, z, m) = \sum_{\substack{n \leq x \\ n \equiv m (\mathrm{mod.}\, k)}} \rho(n, z).$$

Lemma 2. $D(x, z, m) \ll \frac{1}{k}\, x \log x.$

Proof. If $(m, k) = 1$, then the number of the solutions of

$$\lambda d \equiv m \,(\mathrm{mod.}\, k), \quad 1 \leqq \lambda \leqq \left[\frac{x}{d}\right]$$

does not exceed

$$\frac{1}{k}\left[\frac{x}{d}\right] + 1 \leqq \frac{x}{kd} + 1$$

Hence

$$D(x, z, m) \leqq \sum_{\substack{n \leq x \\ n \equiv m (\mathrm{mod.}\, k)}} \sum_{\substack{d \mid n \\ d \leq \frac{x}{k}}} 1 \leqq \sum_{\substack{d \mid n \\ d \leq \frac{x}{k}}} \left(\frac{x}{kd} + 1\right) \ll \frac{1}{k}\, x \log x.$$

Lemma 3. $$\sum_{\substack{n \geq z \\ n \equiv m (\mathrm{mod.}\, k)}} \frac{\rho(n)}{n^{1-\delta}} \ll \frac{1}{k} \frac{\log z}{\delta z^{\delta}} \qquad \text{for } 0 < \delta \leqq 3.$$

Proof. $$\sum_{\substack{n \geq z \\ n \equiv m (\mathrm{mod.}\, k)}} \frac{\rho(n)}{n^{1+\delta}} = \sum_{n \geq z} \frac{D(n) - D(n-1)}{n^{1+\delta}} \leqq \sum_{n \geq z} D(n) \left(\frac{1}{n^{1+\delta}} - \frac{1}{(n+1)^{1+\delta}}\right)$$

$$\leqq (1+\delta) \int_z^{\infty} \frac{D(u)du}{u^{2+\delta}} \ll \frac{1}{k} \int_z^{\infty} \frac{\log u}{u^{1+\delta}}\, du$$

by Lemma 2. By the substitution $u = zv^{\frac{1}{\delta}}$

$$\int_z^{\infty} \frac{\log u}{u^{1+\delta}}\, du = \int_1^{\infty} \frac{\log z}{\delta z^{\delta}\, v^2} dv + \int_1^{\infty} \frac{\log v}{z^{\delta}\, v^2}\, dv \ll \frac{\log z}{\delta z^{\delta}}$$

Lemma 4. $$\sum_{\substack{m, n \geq z \\ n \equiv m (\mathrm{mod.}\, k)}} \frac{\rho(m)\, \rho(n)}{(mn)^{1+\delta}} \ll \frac{1}{k} \frac{\log z}{\delta^3 z^{\delta}} \quad \text{for } 0 < \delta \leqq 3.$$

Proof. Let $d(m)$ be the number of divisors of m. Since

$$\sum_{m \geq z} \frac{\rho(m)}{m^{1+\delta}} \leqq \sum_{m \geq 1} \frac{d(m)}{m^{1+\delta}} = \zeta^2(1+\delta)$$

we have

$$\sum_{\substack{m, n \geq z \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\rho(n)}{(mn)^{1+\delta}} = \sum_{m \geq z} \frac{\rho(m)}{m^{1+\delta}} \sum_{\substack{n \geq z \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(n)}{n^{1+\delta}} \ll \zeta^2(1+\delta\, \frac{\log z}{k \delta z^{\delta}} \ll \frac{1}{k} \frac{\log z}{\delta^3 z^{\delta}}.$$

Lemma 5. $$\sum_{\substack{m<n\leq x \\ n\equiv m (\mathrm{mod},\, k)}} \frac{\rho(m)\rho(n)}{\sqrt{mn}} \ll \frac{1}{k} x \log^2 x.$$

Proof. By the above method, we can easily deduce the result.

Lemma 6. $$\sum_{\substack{m<n\leq x \\ n\equiv m (\mathrm{mod},\, k)}} \frac{\rho(m)\rho(n)}{n-m} \ll \frac{1}{k} x \log^4 x.$$

Proof.

$$\sum_{\substack{m<n\leq x \\ n\equiv m (\mathrm{mod},\, k)}} \frac{\rho(m)\rho(n)}{n-m} = \sum_{lk\leq x} \frac{1}{lk} \sum_{m=1}^{x-lk} \rho(m)\rho(m+lk)$$

$$\leq \frac{1}{k} \sum_{l\leq x} \frac{1}{l} x \log^2 x \sum_{d\mid lk} \frac{1}{d} \ll \frac{1}{k} x \log^4 x$$

by a well known lemma. ([5], Lemma B2.)

Lemma 7. $$\sum_{\substack{m<n\leq x \\ n\equiv m (\mathrm{mod},\, k)}} \frac{\rho(m)\rho(n)}{\sqrt{mn}\log\frac{n}{m}} \ll \frac{1}{k} x \log^4 x.$$

Proof. From the inequality $\dfrac{1}{\sqrt{mn}\log\frac{n}{m}} < \dfrac{1}{\sqrt{mn}} + \dfrac{1}{n-m}$, the result is obvious by Lemma 5 and Lemma 6.

3. **On $K(\sigma, T, z)$.**

For $\sigma>1$, we have

$$f(s, \chi, z) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \sum_{n<z} \mu(n) \chi(n) n^{-s} - 1 = \sum_{n=1}^{\infty} a_n n^{-s}.$$

where $a_n = \chi(n) \sum_{\substack{d\mid n \\ d<z}} \mu(d)$ for $n = 2, 3, \ldots.$ and $a_1 = 0$. Therefore

$$|f(1+\delta+ti, \chi, z)|^2 = \sum_{m,\, n\geq z} \frac{\bar{\chi}(m)\chi(n)}{(mn)^{1+\delta}} \sum_{\substack{d\mid m \\ d<z}} \mu(d) \sum_{\substack{d\mid n \\ d<z}} \mu(d) \left(\frac{m}{n}\right)^{ti}.$$

Hence

$$\sum_{\chi} |f(1+\delta+ti, \chi, z)|^2 = \sum_{\substack{m,\, n\geq z \\ n\equiv m (\mathrm{mod},\, k) \\ (m,\, k)=1}} \frac{\varphi(k)}{(mn)^{1+\delta}} \sum_{\substack{d\mid m \\ d<z}} \mu(d) \sum_{\substack{d\mid n \\ d<z}} \mu(d) \left(\frac{m}{n}\right)^{ti}. \tag{7}$$

For $n>1$

$$\Big|\sum_{\substack{d\mid n \\ d<z}} \mu(d)\Big| = \Big|\sum_{\substack{d\mid n \\ z\leq d\leq n}} \mu(d)\Big| \leq \sum_{\substack{d\mid n \\ z\leq d\leq n}} 1 = \sum_{\substack{d\mid n \\ d\leq\frac{n}{z}}} 1 \qquad (8$$

Lemma 8. If $z\geq k$ and $0<\delta\leq 3$, then $K(1+\delta, T, z) \ll \frac{1}{\delta^4}$. Further if k is sufficiently large, then $K(2, T, z) \leq \frac{1}{2}$.

Proof. From (7) and (8), and by Lemma 4

$$\sum_{\chi} |f(1+\delta+ti, \chi, z)|^2 \ll k \sum_{\substack{m,n\geq z\\ n\equiv m(\mathrm{mod}, k)}} \frac{\rho(m)\rho(n)}{(mn)^{1+\delta}} \ll \frac{\log z}{\delta^3 z^\delta}$$

From this the result is obvious.

For $\sigma=\frac{1}{2}$ we use the inequality

$$|f(\tfrac{1}{2}+ti, \chi, z)|^2 \leqq 2\,(|L(\tfrac{1}{2}+ti, \chi)|^2\,|Q(\tfrac{1}{2}+ti, \chi, z)|^2+1). \tag{9}$$

Since

$$L(s,\chi)=\sum_{\nu=1}^{k}\chi(\nu)\nu^{-s}+\sum_{\nu=1}^{k}\chi(\nu)\,k^{-s}\,(\zeta(s,\nu k^{-1})-(\nu k^{-1})^{-s})$$

we have

$$\mathrm{Max}\ (T-1.5\leqq|t|\leqq T+1.5)\ |L(\tfrac{1}{2}+ti, \chi)| \ll \sqrt{k}T^c. \tag{10}$$

by (2). On the other hand

$$|Q(\tfrac{1}{2}+ti, \chi, z)|^2=\sum_{\substack{m<z\\ n<z}}\frac{\mu(m)\,\mu(n)\,\bar{\chi}(m)\,\chi(n)}{\sqrt{mn}}\left(\frac{m}{n}\right)^{ti}$$

whence

$$\sum_{\chi}|Q(\tfrac{1}{2}+ti, \chi, z)|^2=\varphi(k)\sum_{\substack{m<z,\,n<z\\ n\equiv m(\mathrm{mod}, k)\\ (m,k)=1}}\frac{\mu(m)\mu(n)}{\sqrt{mn}}\left(\frac{m}{n}\right)^{ti}. \tag{11}$$

Lemma 9. If $z\geqq k^2$, then $K(\frac{1}{2}, T, z)\ll kT^{2c}z$.

Proof. We have by (10) and (11)

$$K(\tfrac{1}{2}, T, z)\ll(\sqrt{k}T^c)^2\ \mathrm{Max}\,(T-1.5\leqq|t|\leqq T+1.5)|\sum_{\chi}Q(\tfrac{1}{2}+ti, \chi, z)|^2$$

$$\ll kT^{2c}k\sum_{\substack{m<z,\,n<z\\ n\equiv m(\mathrm{mod}, k)}}\frac{1}{\sqrt{mn}}=k^2T^{2c}\sum_{m<z}\frac{1}{\sqrt{m}}\sum_{\substack{n<z\\ n\equiv m(\mathrm{mod}, k)}}\frac{1}{\sqrt{n}}$$

$$\ll k^2T^{2c}\sum_{m<z}\frac{1}{\sqrt{m}}\left(\frac{\sqrt{z}}{k}+1\right)\ll k\,T^{2c}(z+k\sqrt{z})\ll k\,T^{2c}z$$

Now we write $g(s,\chi,z)=\dfrac{s-1}{s\cos\frac{s}{2T}}f(s,\chi,z)$, $G(s,z)=\sum_{\chi}|g(s,\chi,z)|^2$, and $M(\sigma,z)=\sup_{\Re s=\sigma}G(s,z)$ for $\frac{1}{2}\leqq\sigma\leqq 1+\delta$, $0<\delta\leqq 3$. In this strip, $g(s,\chi,z)$ is regular and satisfies

$$e^{-\frac{|t|}{T}}|f(s,\chi,z)|^2\left|\frac{s-1}{s}\right|^2\ll|g(s\ \chi,z)|^2\ll e^{-\frac{|t|}{T}}|f(s,\chi,z)|^2. \tag{12}$$

Therefore $G(s,z)$ is certainly bounded in this strip for fixed k, T and z.

Lemma 10. If $z=k(T+k)$, then

$$K(\sigma, T, z)\ll\begin{cases}\{k^4T^{4c}(T+k)^2\}^{1-\sigma}\log^4 kT. & (\frac{1}{2}\leqq\sigma\leqq 1)\\ \log^4 kT. & (1\leqq\sigma\leqq 4)\end{cases}$$

Proof. By Lemma 8, Lemma 9 and (12), we have

$$M(1+\delta, z) \ll \frac{1}{\delta^4}, \quad M(\tfrac{1}{2}, z) \ll kT^{2c} z$$

if $z \geqq k^2$. Put $z = k(T+k)$, and use Theorem C, then we get

$$M(\sigma, z) \ll \{k^2 T^{2c} (T+k)\}^{\frac{1+\delta-\sigma}{\frac{1}{2}+\delta}} \left\{\frac{1}{\delta^4}\right\}^{\frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta}}$$

Further we put $\delta = \dfrac{a}{\log kT}$ $(a > 0)$, then we have

$$M(\sigma, z) \ll \{k^2 T^{2c} (T+k)\}^{2(1-\sigma)} (\log^4 kT)^{2\sigma-1}$$

for $\frac{1}{2} \leqq \sigma \leqq 1+\delta$. Thus we get the result by (12).

4. **On I** $(\sigma, \mathrm{T}, \mathrm{z})$.

Lemma 11. If $z \geqq kT$ and $0 < \delta \leqq 1$, then $I(1+\delta, T, z) \ll \dfrac{1}{\delta^5}$.

Proof. From (7) we have

$$\int_{-T}^{T} \sum |f(1+\delta+ti, \chi, z)|^2 \, dt \ll kT \sum_{n \geqq z} \frac{\rho^2(n)}{n^{2+2\delta}} + k \sum_{\substack{n>m \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{(mn)^{1+\delta} \log \frac{n}{m}} \tag{13}$$

if $z \geqq k$. We know from Ingham's estimation ([7], p. 260) that

$$\sum_{n \geqq z} \frac{\rho^2(n)}{n^{2+2\delta}} \leqq \sum_{n \geqq z} \frac{d^2(n)}{n^{2+2\delta}} \ll \frac{1}{\delta^3 z}. \tag{14}$$

On the other hand

$$\sum_{\substack{n>m \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{(mn)^{1+\delta} \log \frac{n}{m}} < \sum_{\substack{n>m \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{(mn)^{1+\delta}} + \sum_{\substack{n>m \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{m^\delta n^{1+\delta} \sqrt{mn} \log \frac{n}{m}}$$

$$< \frac{1}{k\delta^4} + \sum_{\substack{n>m \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{\sqrt{mn} \log \frac{n}{m}} \int_n^\infty \frac{1+\delta}{x^{2+\delta}} \, dx$$

by Lemma 4. Since

$$\sum_{\substack{n>m \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{\sqrt{mn} \log \frac{n}{m}} \int_n^\infty \frac{1+\delta}{x^{2+\delta}} dx = \int_1^\infty \frac{1+\delta}{x^{2+\delta}} \left(\sum_{\substack{m<n \leqq x \\ n \equiv m (\mathrm{mod}, k)}} \frac{\rho(m)\,\rho(n)}{\sqrt{mn} \log \frac{n}{m}} \right) dx$$

$$\ll \frac{1}{k} \int_1^\infty \frac{\log^4 x}{x^{1+\delta}} \, dx \ll \frac{1}{k\delta^5} \tag{15}$$

by Lemma 7. The result is obvious from (13), (14) and (15) if $z \geqq kT$.

Lemma 12. If $z \geqq k^2$, then $I(\frac{1}{2}, T, z) \ll kT^{2c}(kT+z) \log z$.

Proof. From (9) we have

$$\int_{-T}^{T} |f(\tfrac{1}{2}+ti, \chi, z)|^2 dt \ll (\sqrt{k}\,T^c)^2 \int_{-T}^{T} |Q(\tfrac{1}{2}+ti, \chi, z)|^2 dt + T.$$

Further

$$\int_{-T}^{T} \sum_{\chi} |Q(\tfrac{1}{2}+ti, \chi, z)|^2 dt \ll kT \log z + k \sum_{\substack{m<n<z \\ n\equiv m(\mathrm{mod.}\,k)}} \frac{1}{\sqrt{mn}\, \log \frac{n}{m}}$$

$$\ll kT \log z + k\Big(\sum_{\substack{m<n<z \\ n\equiv m(\mathrm{mod.}\,k)}} \frac{1}{\sqrt{mn}} + \sum_{\substack{m<n<z \\ n\equiv m(\mathrm{mod.}\,k)}} \frac{1}{n-m}\Big) \ll kT \log z + z \log z + k\sqrt{z}$$

by (11). Hence we get the result.

We write

$$J(\sigma, z) = \int_{-\infty}^{\infty} G(\sigma+ti, z)\, dt.$$

It is easy to see that

$$J(\sigma, z) \ll \int_0^{\infty} e^{-u} \Big(\int_{-Tu}^{Tu} \sum_{\chi} |f(\sigma+ti, \chi, z)|^2 dt \Big) du. \tag{16}$$

Lemma 13. If $z = k(T+k)$, then

$$I(\sigma, T, z) \ll \{k^4 T^{4c} (T+k)^2\}^{1-\sigma} \log^7 kT. \Big(\frac{1}{2} \leqq \sigma \leqq 1 - \frac{a}{\log kT}\Big)$$

Proof. By Lemma 11, Lemma 12 and (16), we have

$$J(1+\delta, z) \ll \frac{1}{\delta^5}, \quad J(\tfrac{1}{2}, z) \ll kT^{2c}(kT+z) \log z$$

if $z \geqq k(T+k)$. Put $z = k(T+k)$, and use Theorem D, then we get

$$J(\sigma, z) \ll \{k^2 T^{2c} (T+k) \log kT\}^{\frac{1+\delta-\sigma}{\frac{1}{2}+\delta}} \Big\{\frac{1}{\delta^5}\Big\}^{\frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta}}$$

Further we put $\delta = \dfrac{a}{\log kT}$ $(a > 0)$, then we have

$$\begin{aligned} J(\sigma, z) &\ll \{k^2 T^{2c} (T+k) \log kT\}^{2(1-\sigma)} (\log kT)^{5(2\sigma-1)} \\ &\ll \{k^4 T^{4c} (T+k)^2\}^{1-\sigma} \log^5 kT \end{aligned}$$

for $\frac{1}{2} \leqq \sigma \leqq 1$. Thus we get the result by (12).

5. Main theorem.

Theorem 1. If $\frac{1}{2} \leqq \alpha \leqq 1$ and $T \geqq 2$, then

$$N(\alpha, T) \ll \{k^4 T^{4\alpha} (T+k)^2\}^{1-\alpha} \log^8 kT.$$

Proof. Take $\delta = \frac{a}{\log kT}$ $(a>0)$, and insert the results of Lemma 8, Lemma 10, Lemma 11 and Lemma 13 in Lemma 1, we get the above estimation for $\frac{1}{2}+2\delta \leq \alpha < 1$. But if $\frac{1}{2} \leq \alpha \leq \frac{1}{2}+2\delta$, then it is well known that $N(\alpha, T) \ll kT \log kT$. Therefore we obtain the main theorem.

6. Application.

As an application of the main theorem we shall investigate Hoheisel's Problem [7] concerning the difference between consecutive prims in an arithmetic progression.

We know from Hadamard's theory ([8], p. 507.) that if χ is a primitive character with modulus k (>1), then

$$\frac{L'}{L}(s, \chi) = b - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \tag{17}$$

where

$$a = \begin{cases} 0 & \chi(-1) = 1. \\ 1 & \chi(-1) = -1. \end{cases}$$

and $\rho(\chi) = \rho = \beta + \gamma i$ is a typical zero of $L(s, \chi)$ those for which $\gamma = 0$, $\beta \leq 0$ being excluded. If we put $s = 2$ in (17), then we get

$$b + \sum_{|\gamma|<1} \frac{1}{\rho} \ll \log k.$$

Now we consider

$$\frac{1}{2\pi i}\int \frac{x^s}{s}\left(-\frac{L'}{L}(s, \chi)\right) ds$$

where the integral is taken in the positive sence round the rectangle with vertices $\xi \pm Ti$, $\eta \pm Ti$. Take $\xi = 1 + \frac{1}{\log x}$, and $\eta \to -\infty$, then we have

$$\sum_{n \leq x} \chi(n)\, \Lambda(n) + \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + b \ll \frac{x}{T} \log^2 (kx)$$

for $x \geq T \geq 2$, where $\Lambda(n)$ is the Mangoldt function. It is easily seen that the result is valid for the imprimitive character if its leader f is greater than 1. From this we obtain the extended Landau's theorem [9].

Theorem 2. If $x \geq T \geq 2$, then

$$\sum_{n \leq x} \chi(n)\, \Lambda(n) - E(\chi)\, x + \sum_{|\gamma|<T}' \frac{x^\rho}{\rho} + E_1(\chi)\, \frac{x^{\beta_1}}{\beta_1} \ll \frac{x}{T} \log^2 (kx)$$

where

$$E(\chi) = \begin{cases} 1 & \chi = \chi_0 \\ 0 & \chi \neq \chi_0 , \end{cases} \qquad E_1(\chi) = \begin{cases} 1 & \chi = \chi_1 \\ 0 & \chi \neq \chi_1 , \end{cases}$$

χ_0 is the principal character, χ_1 is the special real character in β_1 is the special real zero in Page's theory ([10], Lemma 9), and

$$\sum_{|\gamma|<T}' \frac{x^\rho}{\rho} = \sum_{\substack{|\gamma|<T \\ \rho\neq\beta_1,\, \rho\neq 1-\beta_1}} \frac{x^\rho}{\rho}$$

If we write

$$\psi(x\,;k,\,l) = \sum_{\substack{n\leq x \\ n\equiv l (\mathrm{mod},\, k)}} \Lambda(n)$$

for $(k, l) = 1$, then we have

$$\psi(x\,;k,\,l) = \frac{1}{h}\sum_{n\leq x} \Lambda(n) \sum_{\chi} \chi(n)\, \bar{\chi}(l) = \frac{1}{h} \sum_{\chi} \bar{\chi}(l) \sum_{n\leq x} \chi(n)\, \Lambda(n)$$

$$= \frac{x}{h} - \frac{\bar{\chi}_1(l)}{h} \frac{x^{\beta_1}}{\beta_1} - \frac{1}{h} \sum_{\chi} \bar{\chi}(l) \sum_{\substack{|\gamma|<T \\ \rho=\rho(\chi)}}' \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 (kx)}{T}\right) \tag{18}$$

by Theorem 2.

Since

$$\left|\frac{(x+y)^\rho - x^\rho}{\rho}\right| = \left|\int_x^{x+y} u^{\rho-1}\, du\right| \leqq \int_x^{x+y} u^{\beta-1}\, du \leqq yx^{\beta-1}$$

if $0 < y \leqq x$, we have

$$\psi(x+y\,;k,\,l) - \psi(x\,;k,\,l$$

$$= \frac{y}{h} + E_1(\chi) O\left(\frac{yx^{\beta_1-1}}{h}\right) + \frac{y}{h} O\left(\sum_{|\gamma|<T}' x^{\beta-1}\right) + O\left(\frac{x \log^2 (kx)}{T}\right) \tag{19}$$

It was proved by Tchudakoff [12] that

$$\beta \leqq 1 - \omega(T),\ \omega(T) = \frac{b_1}{\mathrm{Max}\{\log k,\, (\log T)^\mu\}}$$

if $|\gamma| < T$ where b_1 and μ are positive absolute constants. For example we can take $\mu = \frac{4}{5} + \varepsilon$ $(\varepsilon > 0)$ by Titchmarsch [14].

Now, by Theorem 1,

$$\sum_{|\gamma|<T}' x^{\beta-1} = \sum_{|\gamma|<T}' \{x^{-1} + \int_0^\beta x^{\sigma-1} \log x\, d\sigma\} \ll x^{-1} N(0, T) + \int_0^1 \sum_{\substack{|\gamma|<T \\ \beta\geq\sigma}}' x^{\sigma-1} \log x\, d\sigma$$

$$\ll x^{-1}\, kT \log kT + \int_0^1 N(\sigma, T)\, x^{\sigma-1} \log x\, d\sigma$$

$$\ll x^{-1} kT \log kT + \int_0^{1-\omega(T)} \{k^4 T^{4c} (T+k)^2\}^{1-\sigma} (\log^8 kT) x^{\sigma-1} \log x \, dx$$

$$\ll \frac{T}{x} k \log kT + \left\{\frac{k^4 T^{4c} (T+k)^2}{x}\right\}^{\omega(T)} (\log^8 kT) \log x \tag{20}$$

if

$$k^4 T^{4c} (T+k)^2 \leqq x \tag{21}$$

Now we assume that

$$\log k \leqq \frac{1}{20} b_1 \varepsilon \frac{\log x}{\log \log x}. \qquad (0 < \varepsilon < 1) \tag{22}$$

If we take

$$T = x^{\lambda}, \qquad \lambda = \frac{1-\varepsilon}{2+4^c} \tag{23}$$

then

$$k^4 T^{4c} (T+k)^2 \leqq 2k^6 T^{2+4c} \leqq x^{1-\frac{\varepsilon}{2}} \tag{24}$$

when x is sufficiently large, and so (21) is satisfied.

Hence, by (19) and (20)

$$\psi(x+y; k, l) - \psi(x; k, l) = \frac{y}{h} + E_1(\chi) O\left(\frac{y\, x^{\beta_1 - 1}}{h}\right)$$

$$+ \frac{y}{h} O\left\{\frac{T}{x} k \log kT + \exp\left(\omega(T) \log \frac{k^4 T^{4c} (T+k)^2}{x}\right) (\log^8 kT) \log x\right\}$$

$$+ \frac{y}{h} O\left(\frac{hx \log^2 (kx)}{yT}\right)$$

Using (22), (23) and (24), we have, when x is sufficiently large,

$$\frac{T}{x} k \log kT \ll \exp\left\{-(1-\lambda) \log x + \frac{1}{20} b_1 \varepsilon \frac{\log x}{\log \log x} + \log \log x\right\}$$

$$\ll \exp\left\{-\left(1 - \frac{3}{2}\lambda\right) \log x\right\},$$

$$\exp\left\{\omega(T) \log \frac{k^4 T^{4c} (T+k)^2}{x}\right\} (\log^8 kT) \log x$$

$$\ll \exp\left\{9 \log \log x - \omega(T) \frac{\varepsilon}{2} \log x\right\},$$

$$\frac{hx \log^2 (kx)}{yT} \ll \exp\left\{(1-\lambda) \log x + \frac{1}{20} b_1 \varepsilon \frac{\log x}{\log \log x} + 2 \log \log x - \log y\right\}$$

$$\ll \exp\left\{-\frac{1}{4} \varepsilon \log x\right\}$$

if $y \geqq x^{\frac{1+4c}{2+4c} + \varepsilon} \geqq x^{1-\lambda+\frac{\varepsilon}{2}}$.

Hence we obtain

$$\psi(x+y;k,l)-\psi(x;k,l)=\frac{y}{h}+E_1(\chi)O\left(\frac{yx^{\beta_1-1}}{h}\right)+$$
$$+\frac{y}{h}O\left\{\exp\left(9\log\log x-\omega(T)\frac{\varepsilon}{2}\log x\right)\right\}+\frac{y}{h}O\{\exp(-b_2\log x)\}. \quad (25)$$

where $b_2=b_2(\varepsilon)$ is a positive constant.

Now

$$9\log\log x-\omega(T)\frac{\varepsilon}{2}\log x\leqq 9\log\log x-\frac{1}{2}b_1\,\varepsilon\frac{\log x}{\log k}\leqq-\frac{1}{20}b_1\,\varepsilon\frac{\log x}{\log k}$$

if $\lambda^{\mu}(\log x)^{\mu}\leqq\log k\leqq\frac{b_1\,\varepsilon}{20}\frac{\log x}{\log\log x}$. On the other hand

$$9\log\log x-\omega(T)\frac{\varepsilon}{2}\log x\leqq 9\log\log x-\frac{b_1\,\varepsilon\log x}{2\lambda^{\mu}(\log x)^{\mu}}\leqq-b_3(\log x)^{1-\mu}$$

if $\log k\leqq\lambda^{\mu}(\log x)^{\mu}$ and x is sufficiently large where $b_3=b_3(\varepsilon)$ is a positive constant. Thus we conclude the following Lemma.

Lemma 14. If $(l,k)=1$ and $x^{\frac{1+4c}{2+4c}+\varepsilon}\leqq y\leqq x$ $(\varepsilon>0)$, then there exist positive constants $A=A(\varepsilon)$, $B=B(\varepsilon)$ and $C=C(\varepsilon)$ such that

$$\psi(x+y;k,l)-\psi(x;k,l)=\frac{y}{h}+E_1(\chi)O\left(\frac{yx^{\beta_1-1}}{h}\right)+$$
$$+\begin{cases}O\left\{\frac{y}{h}\exp\left(-A\frac{\log x}{\log k}\right)\right\}. & C(\log x)^{\mu}\leqq\log k\leqq B\frac{\log x}{\log\log x}\\ O\left\{\frac{y}{h}\exp\left(-A(\log x)^{1-\mu}\right)\right\}. & \log k\leqq C(\log x)^{\mu}\end{cases}$$

From this we can prove the following theorem by usual method.

Theorem 3. If $(k,l)=1$, $x^{\frac{1+4c}{2+4c}+\varepsilon}\leqq y\leqq x$ and $\pi(x;k,l)$ denotes the number of primes satisfying $p\leqq x$, $p\equiv l$ (mod. k), then

$$\pi(x+y;k,l)-\pi(x;k,l)=\frac{1}{h}\int_x^{x+y}\frac{du}{\log u}+O\left(\frac{yx^{\beta_1-1}}{h\log x}\right)+$$
$$+\begin{cases}O\left\{\frac{y}{h}\exp\left(-A\frac{\log x}{\log k}\right)\right\}. & C(\log x)^{\mu}\leqq\log k\leqq B\frac{\log x}{\log\log x}\\ O\left\{\frac{y}{h}\exp\left(-A(\log x)^{1-\mu}\right)\right\}. & \log k\leqq C(\log x)^{\mu}\end{cases}$$

Further if we use Siegel's theorem ([2], [15]), we obtain the following theorem.

Theorem 4. If $(k,l)=1$, $x^{\frac{1+4c}{2+4c}+\varepsilon}\leqq y\leqq x$ and $\log k\leqq\delta\log\log x$ $(\delta>0)$ then

$$\pi(x+y; k, l)-\pi(x; k, l) = \frac{1}{h}\int_x^{x+y}\frac{du}{\log u} + O\left\{\frac{y}{h}\exp\left(-A(\log x)^{1-\mu}\right)\right\}.$$

where $A = A(\varepsilon, \delta)$ is a positive constant.

Finally we get the following result. We omit the proof. ([1], [15]).

Theorem 5. Suppose that a and m are the given positive numbers. If N is an odd integer and $N \geqq A$, then there exist such primes p, p' and p'' that

$$N = p + p' + p'', \ \frac{N}{3} - a\frac{N}{(\log N)^m} < p, p', p'' < \frac{N}{3} + a\frac{N}{(\log N)^m}$$

where $A = A(a, m)$ is a positive constant.

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