1. *On Riemannian Spaces Admitting a Family of Totally Umbilical Hypersurfaces. I.*

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§1. Let V_n be an n-dimensional Riemannian space with the fundamental tensor $g_{\lambda\mu}$ $(\lambda,\mu,\nu,\ldots = 1, 2,\ldots, n)$ and assume that there exists a family of totally umbilical hypersurfaces

$$\sigma(x^\lambda) = \text{const.}. \tag{1.1}$$

If we denote the parametric representation of its hypersurfaces by $x^\lambda = x^\lambda(x^i)$ $(i,j,k,\ldots = \dot{1},\dot{2},\ldots,\dot{n}-\dot{1})$, then from (1.1) we have by differentiation with respect to x^i

$$\sigma_\lambda B_i^{\cdot\lambda} = 0,$$

where $\sigma_\lambda = \dfrac{\partial\sigma}{\partial x^\lambda}$, $B_i^{\cdot\lambda} = \dfrac{\partial x^\lambda}{\partial x^i}$. Furthermore, differentiating with respect to x^j, we have

$$\sigma_{\lambda;\mu} B_i^{\cdot\lambda} B_j^{\cdot\mu} + \sigma_\lambda H_{ij}^{\cdot\cdot\lambda} = 0,$$

where $H_{ij}^{\cdot\cdot\lambda}$ is an Euler-Schouten's curvature tensor. If we denote the fundamental tensor and normals of the hypersurfaces by g_{ij} and B^λ respectively, we have, because of $H_{ij}^{\cdot\cdot\lambda} = Hg_{ij}B^\lambda$,

$$\sigma_{\lambda;\mu} B_i^{\cdot\lambda} B_j^{\cdot\mu} + H\sigma_\lambda B^\lambda g_{ij} = 0,$$

from which follows

$$(\sigma_{\lambda;\mu} + H\sigma_\nu B^\nu g_{\lambda\mu}) B_j^{\cdot\lambda} B_j^{\cdot\mu} = 0.$$

Consequently $\sigma_{\lambda;\mu}$ must take the form

$$\sigma_{\lambda;\mu} = \rho\, g_{\lambda\mu} + v_\lambda \sigma_\mu + v_\mu \sigma_\lambda\,, \tag{1.2}$$

where $\rho = -H\sigma_\nu B^\nu$ and v_λ is a certain vector.

Conversely, if (1.2) holds, we know easily that the hypersurfaces $\sigma(x^\lambda) = \text{const.}$ are totally umbilical.

Differentiating (1.2) and substituting the resulted equations in Ricci identities $\sigma_{\lambda;\mu\nu} - \sigma_{\lambda;\nu\mu} = -\sigma_\omega R^{\omega}_{\cdot\lambda\mu\nu}$, we have

$$\begin{aligned} -\sigma_\omega R^{\omega}_{\cdot\lambda\mu\nu} = {} & \{(\rho_\nu - \rho v_\nu)\, g_{\lambda\mu} - (\rho_\mu - \rho v_\mu)\, g_{\lambda\nu}\} \\ & + \{(v_{\lambda;\nu} - v_\lambda v_\nu)\, \sigma_\mu - (v_{\lambda;\mu} - v_\lambda v_\mu)\, \sigma_\nu\} + \sigma_\lambda\, (v_{\mu;\nu} - v_{\nu;\mu})\,. \end{aligned} \tag{1.3}$$

If we put $\sigma_\lambda = \sqrt{\sigma^\mu\sigma_\mu}\, B_\lambda$, where $\sigma^\mu\sigma_\mu = g^{\mu\nu}\sigma_\mu\sigma_\nu$ and $B_\lambda = g_{\lambda\nu}B^\nu$, we have from (1.3)

$$(1.4)\qquad B_\omega B^\nu B_j^{\cdot\lambda} B_k^{\cdot\mu} R^{\omega}_{\cdot\lambda\mu\nu} = \frac{-1}{\sigma^\mu\sigma_\mu}\sigma^\nu(\rho_\nu-\rho\, v_\nu)\, g_{jk} + B_j^{\cdot\lambda} B_k^{\cdot\mu}(v_{\lambda;\mu}-v_\lambda v_\mu).$$

On the other hand, according to Gauss equations, we have

$$R^i_{\cdot jkh} = B^{i\mu\nu\omega}_{\lambda jkh} R^\lambda_{\cdot\mu\nu\omega} + H^2(g_{jk}\delta^i_h - g_{jh}\delta^i_k),$$

where $R^i_{\cdot jkh}$ is the curvature tensor of the hypersurfaces and $B^{i\mu\nu\omega}_{\lambda jkh} = B^i_{\cdot\lambda} B_j^{\cdot\mu} B_k^{\cdot\nu} B_h^{\cdot\omega}$. Summing for i and h, we have

$$\begin{aligned} R_{jk} &= (\delta^\omega_\lambda - B_\lambda B^\omega) B_j^{\cdot\mu} B_k^{\cdot\nu} R^\lambda_{\cdot\mu\nu\omega} + (n-2)H^2 g_{jk} \\ &= B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\mu\nu} - B_\lambda B^\omega B_j^{\cdot\mu} B_k^{\cdot\nu} R^\lambda_{\cdot\mu\nu\omega} + (n-2)H^2 g_{jk}. \end{aligned}$$

Substituting (1.4), we obtain

$$(1.5)\qquad \begin{aligned} R_{jk} = {} & B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\mu\nu} - B_j^{\cdot\lambda} B_k^{\cdot\mu}(v_{\lambda;\mu} - v_\lambda v_\mu) \\ & + \left\{(n-2)H^2 + \frac{1}{\sigma^\mu\sigma_\mu}\sigma^\nu(\rho_\nu - \rho\, v_\nu)\right\} g_{jk}. \end{aligned}$$

Putting $v_\lambda B_j^{\cdot\lambda} = v_j$ and differentiating with respect to x^k, we have

$$v_{\lambda;\mu} B_j^{\cdot\lambda} B_k^{\cdot\mu} + v_\lambda H_{jk}^{\cdot\cdot\lambda} = v_{j;k},$$

from which follows

$$v_{\lambda;\mu} B_j^{\cdot\lambda} B_k^{\cdot\mu} = v_{j;k} - v_\lambda B^\lambda H g_{jk}.$$

Thus (1.5) takes the form

$$(1.6)\qquad R_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\mu\nu} + v_j v_k - v_{j;k} + \beta g_{jk}.$$

Since $v_{j;k} = v_{k;j}$, we find that v_j is a gradient vector.

Now we put

$$\Pi_{\lambda\mu} = -\frac{R_{\lambda\mu}}{n-2} + \frac{R g_{\lambda\mu}}{2(n-1)(n-2)}$$

and assume that $\Pi_{\lambda\mu}$ takes the form

$$(1.7)\qquad \Pi_{\lambda\mu} = u g_{\lambda\mu} + \zeta_\lambda \sigma_\mu + \zeta_\mu \sigma_\lambda,$$

where u is a scalar function of x^λ and ζ_λ a certain vector. In this case, directions orthogonal to the vectors σ_λ and ζ_λ are Ricci principal directions. Substituting $R_{\lambda\mu}$ obtained from (1.7) in (1.6), we have the equations of the form

$$(1.8)\qquad R_{jk} = \gamma g_{jk} + v_j v_k - v_{j;k}.$$

Thus we have

Theorem 1.1. In order that the tensor $\Pi_{\lambda\mu}$ of a space admitting a family of ∞^1 totally umbilical hypersurfaces $\sigma(x^\lambda)$ =const. takes the form

$$\Pi_{\lambda\mu} = ug_{\lambda\mu} + \zeta_\lambda\sigma_\mu + \zeta_\mu\sigma_\lambda \qquad \left(\sigma_\lambda = \frac{\partial\sigma}{\partial x^\lambda}\right),$$

it is necessary and sufficient that the Ricci tensors of the hypersurfaces take the form

$$R_{jk} = \gamma g_{jk} + v_j v_k - v_{j;k},$$

where v_j is a certain gradient vector.

Especially when tangential directions of the hypersurfaces are all Ricci directions, (1.7) takes the form

$$\Pi_{\lambda\mu} = ug_{\lambda\mu} + \kappa\sigma_\lambda\sigma_\mu . \tag{1.9}$$

Thus we have

Cor. 1. If tangential directions of the totally umbilical hypersurfaces $\sigma(x^\lambda)$ =const. are Ricci principal directions, then (1.8) holds.

Cor. 2.[2] In an Einstein space admitting totally umbilical hypersurfaces $\sigma(x^\lambda)$ =const. (1.8) holds.

§ 2. Assuming that (1.7) holds, we shall calculate the scalar curvature $\bar{R}$ of the totally umbilical hypersurfaces $\sigma(x^\lambda)$ =const.. From (1.5) we have

$$\begin{aligned}\bar{R} = g^{jk} R_{jk} = g^{jk}B_j^{\cdot\mu}B_k^{\cdot\nu}R_{\mu\nu} - (g^{\lambda\mu} - B^\lambda B^\mu)(v_{\lambda;\mu} - v_\lambda v_\mu) \\ + (n-1)\{(n-2)H^2 + \frac{1}{\sigma^\mu\sigma_\mu}\sigma^\nu(\rho_\nu - \rho v_\nu)\}.\end{aligned} \tag{2.1}$$

Since we have from (1.7)

$$R_{\mu\nu} = \left\{\frac{R}{2(n-1)} - (n-2)u\right\}g_{\mu\nu} - (n-2)(\zeta_\mu\sigma_\nu + \zeta_\nu\sigma_\mu),$$

we obtain

$$g^{jk}B_j^{\cdot\mu}B_k^{\cdot\nu}R_{\mu\nu} = \frac{R}{2} - (n-1)(n-2)u . \tag{2.2}$$

Moreover, from (1.3) we have

$$\begin{aligned}-\sigma_\omega R^{\omega}_{\cdot\cdot\nu} = (n-1)(\rho_\nu - \rho v_\nu) + \sigma^\lambda(v_{\lambda;\nu} - v_\lambda v_\nu) - g^{\lambda\mu}(v_{\lambda;\mu} - v_\lambda v_\mu)\sigma_\nu \\ + \sigma^\lambda(v_{\lambda;\nu} - v_{\nu;\lambda}).\end{aligned}$$

Multiplying by σ^ν and summing for ν, we obtain

$$-\sigma^\omega\sigma^\nu R_{\omega\nu} = (n-1)\sigma^\nu(\rho_\nu - \rho v_\nu) - \sigma^\nu\sigma_\nu(g^{\lambda\mu} - B^\lambda B^\mu)(v_{\lambda;\mu} - v_\lambda v_\mu). \tag{2.3}$$

However, because of (1.7), we have

$$\sigma^{\omega}\sigma^{\nu}R_{\omega\nu}=\left\{\frac{R}{2(n-1)}-(n-2)u-2(n-2)\sigma^{\mu}\zeta_{\mu}\right\}\sigma^{\nu}\sigma_{\nu}.$$

On the other hand, multiplying (1.7) by $g^{\lambda\mu}$ and summing for λ and μ, we have

$$-\frac{R}{2(n-1)}=nu+2\sigma^{\mu}\zeta_{\mu}.$$

Thus we have

$$\sigma^{\omega}\sigma^{\nu}R_{\omega\nu}=\left(\frac{R}{2}+(n-1)(n-2)u\right)\sigma^{\nu}\sigma_{\nu}.$$

Substituting in (2.3), we have

$$(2.4)\qquad -(g^{\lambda\mu}-B^{\lambda}B^{\mu})(v_{\lambda;\mu}-v_{\lambda}v_{\mu})+\frac{n-1}{\sigma^{\mu}\sigma_{\mu}}\sigma^{\nu}(\rho_{\nu}-\rho v_{\nu})$$
$$=-\frac{R}{2}-(n-1)(n-2)u.$$

Substituting (2.2) and (2.4) in (2.1), we obtain

$$\bar{R}=(n-1)(n-2)(-2u+H^2).$$

When V_n is an Einstein space, (1.7) becomes

$$(2.5)\qquad \Pi_{\lambda\mu}=u\,g_{\lambda\mu}$$

and consequently $u=-\dfrac{R}{2n(n-1)}$, from which follows

$$\bar{R}=(n-1)(n-2)\left(\frac{R}{n(n-1)}+H^2\right).$$

If $n\geq 3$, R=const.. Since normals of the hypersurfaces σ =const. are Ricci directions, also H=const.. Thus we have

Theorem 2.1.[2] In an Einstein space admitting a family of totally umbilical hypersurfaces, the mean curvature and scalar curvature of the hypersurfaces are constant on the hypersurfaces.

§ 3. From the theorem 1.1 we have readily

Theorem 3.1. In a space admitting a family of totally umbilical hypersurfaces $\sigma(x^{\lambda})$ =const., if the tensor $\Pi_{\lambda\mu}$ takes the form (1.7)

$$\Pi_{\lambda\mu}=ug_{\lambda\mu}+\zeta_{\lambda}\sigma_{\mu}+\zeta_{\mu}\sigma_{\lambda}$$

and the hypersurfaces $\sigma(x^{\lambda})$ =const. are all Einstein spaces, the equations of the form

$$v_{i;j}=a\,g_{ij}+v_iv_j$$

hold, that is to say, the hypersurfaces admit a concircular transformation.[3]

When we replace (1.7) by (1.9)

$$\Pi_{\lambda\mu} = u\, g_{\lambda\mu} + \kappa\sigma_\lambda\sigma_\mu$$

and also by (2.5)[2]

$$\Pi_{\lambda\mu} = u\, g_{\lambda\mu} \qquad (V_n \text{ is an Einstein space}),$$

the theorem holds.

Next we consider the case when the totally umbilical hypersurfaces σ =const. are conformally flat. We assume $n>3$ and put

$$\Pi_{jk} = -\frac{R_{jk}}{n-3} + \frac{\bar{R}\, g_{jk}}{2(n-2)(n-3)}.$$

From (1.8) we have

$$(n-3)\Pi_{jk} = v_{j;k} - v_j v_k + \tau\, g_{jk}, \tag{3.1}$$

where τ is a certain scalar. Since

$$\Pi_{j[k;l]} = 0, \qquad v_{j;kl} - v_{j;lk} = -v_m R^m_{\cdot jkl},$$

we have

$$-v_m R^m_{\cdot jkl} - (v_{j;l}v_k - v_{j;k}v_l) + (\tau_l\, g_{jk} - \tau_k g_{jl}) = 0. \tag{3.2}$$

However

$$\begin{aligned} v_m R^m_{\cdot jkl} &= v_m(-\Pi_{jk}\,\delta^m_l + \Pi_{jl}\,\delta^m_k - g_{jk}\,\Pi^m_{\cdot l} + g_{jl}\,\Pi^m_{\cdot k}) \\ &= -\Pi_{jk}\,v_l + \Pi_{jl}\,v_k - g_{jk}\,v_m\Pi^m_{\cdot l} + g_{jl}\,v_m\Pi^m_{\cdot k}. \end{aligned}$$

Substituting (3.1), we have

$$\begin{aligned} v_m R^m_{\cdot jkl} = -\frac{1}{n-3}\{&(v_{j;k}v_l - v_{j;l}v_k) + g_{jk}\,(2\tau v_l - v^m v_m v_l \\ &+ v^m v_{m;l}) - g_{jl}\,(2\tau v_k - v^m v_m v_k + v^m v_{m;k})\}. \end{aligned}$$

Consequently (3.2) becomes

$$\begin{aligned} (n-2)\,(v_{j;l}v_k - v_{j;k}v_l) + \{&(n-3)\tau_k + 2\tau v_k - v^m v_m v_k \\ &+ v^m v_{m;k}\}g_{jl} - \{(n-3)\tau_l + 2\tau v_l - v^m v_m v_l + v^m v_{m;l}\}g_{jk} = 0. \end{aligned} \tag{3.3}$$

Multiplying by g^{jl} and summing for j and l, we have

$$(g^{jl}v_{j;l} + 2\,\tau - v^m v_m)\,v_k + (n-3)\,\tau_k = 0. \tag{3.4}$$

Multiplying (3.3) by v^j and summing for j, we have

$$(v^j v_{j;l}v_k - v^j v_{j;k}v_l) + (\tau_k v_l - \tau_l v_k) = 0. \tag{3.5}$$

From (3.4) and (3.5) we find that τ_k and $v^j v_{j;k}$ are proportional to v_k. Hence from (3.3) we get the equations of the form

$$v_{j;k} = a\, g_{jk} + b\, v_j\, v_k.$$

Consequently (3.1) takes the form

$$II_{jk} = p\, g_{jk} + q\, v_j\, v_k\,,$$

from which follows that v^j represents Ricci directions, namely the hypersurfaces σ =const. admit concircular tranasformations.

Since a conformally flat space admitting a concircular transformation is a subprojective space of B. Kagan,[1), 4)] we get

Theorem 3.2. In a space admitting a family of totally umbilical hypersurfaces $\sigma(x^\lambda)$ =const., if the tensor $II_{\lambda\mu}$ takes the form (1.7)

$$II_{\lambda\mu} = u\, g_{\lambda\mu} + \zeta_\lambda \sigma_\mu + \zeta_\mu \sigma_\lambda$$

and the hypersurfaces are conformally flat, then these hypersurfaces are subprojective in the sense of Kagan ($n > 3$).

When we replace (1.7) by (1.9) and also when V_n is an Einstein space,[2)] the theorem holds.

References.

1) T. Adati: On Subprojective Spaces. Not yet published.

2) Y.C. Wong: Family of totally umbilical hypersurfaces in an Einstein space. Annals of Math., 44 (1943), 271-297.

3) K. Yano: Concircular Geometry I. Proc. Imp. Acad. Tokyo, 16 (1940) 195-200.

4) K. Yano and T. Adati: On certain spaces admitting concircular transformation. To be published in Proc. Jap. Acad.