14. *On the Type of an Open Riemann Surface.*

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1. Let F be an open abstract Riemann surface and Γ be its ideal boundary. Suppose that $F_n (n = 0, 1, \dots)$ is the relatively compact (shlicht or non) subdomain of F satisfying the following four conditions:

1°) F_o is simply and $F_n (n \neq 0)$ is finitely connected,

2°) $F_n \subset F_{n+1}$,

3°) if Γ_n is the boundary of F_n, Γ_n consists of a finite number of analytic closed curves and $\Gamma_n \frown \Gamma_{n+1} = 0$,

4°) $\bigcup\limits_{n=0}^{\infty} F_n = F$.

Putting $R_n = F_n - \bar{F}_o$, the boundary of R_n consists of Γ_n and Γ_o. Let P be the inner point of R_n and denote by $\omega_n = \omega_n (\Gamma_n, P, R_n)$ the harmonic measure of Γ_n at P with respect to the domain R_n. Then we call

$$D(\omega_n) = \iint\limits_{R_n} \left[\left(\frac{d\omega_n}{dx}\right)^2 + \left(\frac{d\omega_n}{dy}\right)^2\right] dxdy, \qquad t = x + iy,$$

the Dirichlet integral of ω_n with respect to the domain R_n, where t is the local parameter.

R. Nevanlinna [3] has proved the following:

Theorem. The ideal boundary Γ of the Riemann surface F is of harmonic measure zero if and only if $\lim\limits_{n\to\infty} D(\omega_n) = 0$.

2. Let u be the harmonic function in the domain R_n such that

$$u = \begin{cases} 0 & \text{on } \Gamma_o, \\ \log \mu_n & \text{on } \Gamma_n \qquad (\mu_n > 1), \end{cases}$$

and, if v is the conjugate harmonic function of u, then the total variation on Γ_o equals to 2π, i.e.,

$$\int\limits_{\Gamma_o} dv = 2\pi .$$

In this case we call $\log \mu_n$ the modul of the domain R_n.

We shall show the following:

Theorem 1. Let $\log \mu_n$ be the modul of R_n and ω_n be the harmonic measure of Γ_n with respect to R_n. Then we have

$$\log \mu_n = \frac{2\pi}{D(\omega_n)}.$$

Proof. Let $\bar{\omega}_n$ be the conjugate harmonic function of ω_n. Then it is easily seen that

$$D(\omega_n) = \int_{\Gamma_o} d\bar{\omega}_n = \int_{\Gamma_o} \frac{d\omega_n}{d\nu} ds,$$

where ν is the outer normal, ds is the line-element of Γ_o and the integral is taken in the positive sense of Γ_o with respect to F_o.

We consider the harmonic function

$$u = \frac{2\pi}{D(\omega_n)} \omega_n$$

and denote its conjugate harmonic function by v. Then we get

$$\int_{\Gamma_o} dv = \int_{\Gamma_o} \frac{du}{d\nu} ds = \frac{2\pi}{D(\omega_n)} \int_{\Gamma_o} \frac{d\omega_n}{d\nu} ds = 2\pi.$$

Since $u = 0$ on Γ_o and $u = \frac{2\pi}{D(\omega_n)}$ on Γ_n, we obtain, from the definition of the modul,

$$\log \mu_n = \frac{2\pi}{D(\omega_n)}. \qquad \text{(q.e.d.)}$$

From this and Nevanlinna's theorem we can easily show the following:

Theorem 2. The ideal boundary Γ of the Riemann surface F is of harmonic measure zero if auy only if $\lim_{n\to\infty} \mu_n = \infty$.

Remark. A. Pfluger [4] has shown only the sufficiency of this condition.

3. We suppose that we can define a conformal metric on F by the line-element

$$ds = \lambda(t)\,|dt|,$$

where $t = x + iy$ is the local parameter and $\lambda(t)$ a single-valued, positive function of t. We can suppose here that the line-element ds is invariant for any conformal transformation and the distances between a fixed point P on F and ideal boundary points of F are infinite and finally that the distances between P and inner points of F are finite.

By F_ρ we denote the set of points such that the distance between any point of F_ρ and the point P is less than ρ $(0 < \rho < \infty)$ and by Γ_ρ the boundary of F_ρ and further by $L(\rho)$ the length of

Γ_ρ defined by the line-element ds.

We put $R_\rho = F_\rho - \bar{F}_1 (\rho > 1)$. Denote by $\log \mu_\rho$ the modul of the domain R_ρ and by u the harmonic function in R_ρ such that

$$u = \begin{cases} 0 & \text{on } \Gamma_0, \\ \log \mu_\rho & \text{on } \Gamma_\rho. \end{cases}$$

If v represents the conjugate harmonic function of u, then we have

$$2\pi = \int_{\Gamma_\rho} dv = \int_{\Gamma_\rho} \frac{du}{d\nu} ds,$$

where the integral is taken in the positive sense of Γ_ρ with respect to the domain R_ρ.

Let $D(\rho)$ be the Dirichlet integral of u with respect to R_ρ. Then it is immediate that

$$D(\rho) = \iint_{R_\rho} \left[\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 \right] dxdy = \int_{\Gamma_\rho} u dv = 2\pi \log \mu_\rho .$$

Therefore, using $\int_{\Gamma_\rho} ds = L(\rho)$ and the Schwarz inequality, wet get

$$4\pi^2 \leqq \int_{\Gamma_\rho} ds \int_{\Gamma_\rho} \left(\frac{du}{d\nu}\right)^2 ds \leqq L(\rho) \frac{dD(\rho)}{d\rho},$$

whence

$$4\pi^2 \frac{d\rho}{L(\rho)} \leqq dD(\rho) .$$

Integrating the both sides from ρ_0 to ρ, we obtain

$$4\pi^2 \int_{\rho_0}^{\rho} \frac{d\rho}{L(\rho)} \leqq D(\rho) - D(\rho_0) = 2\pi \log \frac{\mu_\rho}{\mu_{\rho_0}} .$$

Thus, from Theorem 2, we have the following:

Theorem 3. (Laasonen [3]). *If the integral* $\int^\infty \frac{d\rho}{L(\rho)}$ *diverges, the ideal boundary of the Riemann surface is of harmonic measure zero.*

4. Let us consider the case that the Riemann surface F is simply connected. In this case, F is mapped conformally on the finite or infinite circle in the complex z-plane. Hence we can suppose that F_ρ is the image of the circle $|z| < \rho$ on F as in § 3. If we put $R_\rho = F_\rho - \bar{F}_1$ $(\rho > 1)$, then the modul $\log \mu_\rho$ of R_ρ equals to $\log \rho$. Moreover, we can take

$$ds = |z'(w)|\,|dw| = |dz|$$

as the conformal metric on F, where $z(w)$ is the mapping function of F into the z-plane. Since we get

$$L(\rho) = 2\pi\rho\,,$$

we have

$$(*)\qquad \int_1^\rho \frac{d\rho}{L(\rho)} = \frac{1}{2\pi}\log\rho = \frac{1}{2\pi}\log\mu_\rho\,.$$

Thus the following theorem is obtained:

Theorem 4. (Ahlfors [1]). *In order that the simply connected open Riemann surface is of the parabolic type, it is necessary and sufficient that there exists a metric ds on F such that the integral*

$$\int^\infty \frac{d\rho}{L(\rho)}$$

diverges.

Proof. Sufficiency of this condition is obtained without difficulty from Theorem 3. Necessity is obvious from (*).

References.

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