30. Notes on Fourier Analysis (XL): Remark on the Rademacher System.

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§ 1. Let $\{r_n(x)\}$ denote the Rademacher system, and let $\{p_n\}$ $(n=1, 2, \ldots)$ be an increasing sequence of positive numbers. If we denote $P_n = p_1+p_2+\cdots+p_n$ and

$$\varphi_n(x) = [p_1r_1(x)+p_2r_2(x)+\cdots+p_nr_n(x)]/P_n \tag{1}$$

$(n=1, 2, \ldots)$, the following theorems are known (J. D. Hill [1]):

(i) *The set of convergence points of* $\varphi_n(x)$ *is of measure* 0 *or* 1[1]; (ii) *if the series*

$$\sum_{n=1}^{\infty} (p_n/P_n)^2 \tag{2}$$

converges, then $\varphi_n(x)$ *converges to zero almost everywhere; and conversely* (iii) *if* $\varphi_n(x)$ *converges in a set of positive measure, its limit is necessarily zero almost everywhere, and moreover*

$$\lim_{n\to\infty} p_n/P_n = 0.\text{[2]} \tag{3}$$

Let us consider now the condition which implies the convergence almost everywhere of $\varphi_n(x)$. It is also known that the condition (3) is insufficient to assert such convergence (G. Maruyama [4] and the author [6]). In this note, by determining the decreasing order of (3), we shall give a sufficient condition different from the convergence of (2)[3].

THEOREM. *If we have*

$$p_n/P_n = o(1/\log\log P_n) \qquad \text{as } n\to\infty, \tag{4}$$

then $\varphi_n(x)$ *converges to zero almost everywhere.*

The condition (4) is the best possible one of this form, in fact, there exists an increasing sequence of positive numbers $\{p_n\}$ such that $p_n/P_n = O(1/\log\log P_n)$ as $n\to\infty$, and $\varphi_n(x)$ diverges almost everywhere. An example with this property was furnished by Mr.

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1) We shall understand, throughout this paper, that the sets are included in (0, 1), that is, $0<x<1$.

2) Cf. Remark 3, § 3.

3) Cf. Remark 4, § 3.

G. Maruyama [4]: Let $p_1 = 1$ and $p_n = \exp(n/\log n)/\log n (n \geqq 2)$[4], then by easy calculation we have $P_n \sim \exp(n/\log n)$, $\log\log P_n \sim \log n$ and $p_n/P_n \sim 1/\log\log P_n$; and as he proved the divergence almost everywhere of $\varphi_n(x)$ may be shown using the Kolmogoroff lemma on the law of the iterated logarithm.

§ 2. PROOF OF THEOREM. As P_n tends to the infinity with n, we can choose an integer n_1 such that

(5) $$P_{n_1} > 1, \text{ and } p_n/P_n < 1/3 \quad \text{for } n \geqq n_1$$

in virtue of (4). We shall define a sequence of integers $\{n_k\}$ by induction. If $n_1, n_2, \dots, n_{k-1}$ are defined, we can find an integer n_k such that

(6) $$P_{n_{k-1}} < P_{n_k} \leqq 2P_{n_{k-1}} \text{ and } P_{n_k+1} > 2P_{n_{k-1}};$$

this possibility may be easily conceived from the relation:

$$P_{n_{k-1}+1}/P_{n_{k-1}} = 1 + (p_{n_{k-1}+1}/P_{n_{k-1}+1})/[1-(p_{n_{k-1}+1}/P_{n_{k-1}+1})] < 1 + (1/3)/(1-1/3) < 2 \qquad (k \geqq 2).$$

The sequence $\{n_k\}$ is thus defined. Let us put

$$S_n(x) = p_1 r_1(x) + \dots + p_n r_n(x),\ S_n^*(x) = \max_{1 \leq m \leq n} |S_m(x)| \quad (n = 1, 2, \dots).$$

For a given $\delta > 0$, denote by $E_k (k = 1, 2, \dots)$ the set of all x such that $|S_n(x)| > \delta P_n$ for at least one value of n, $n_{k-1} < n \leqq n_k$, and put

$$M_k = \mathop{\mathrm{E}}_x [\,|S_{n_k}^*(x)| > \delta P_{n_{k-1}}] \qquad (k = 2, 3, \dots).$$

If the series $\sum_{k=1}^{\infty} |E_k|$ converges for every $\delta > 0$, we may complete the proof in virtue of the well known Borel-Cantelli theorem, hence it is sufficient to prove the convergence of the series

(7) $$\sum_{k=2}^{\infty} |M_k|,$$

since $E_k \subset M_k (k = 2, 3, \dots)$. From the Marcinkiewicz-Zygmund inequality[5]

(8) $$\int_0^1 \exp(aS_n^*(x))dx \leqq 32 \exp\left(\frac{1}{2}a^2 B_n\right)$$

where $a = a_n > 0$ and $B_n = p_1^2 + \dots + p_n^2 (n = 1, 2, \dots)$, we have

4) This definition is different from his in its form, but for our purpose these two are essentially the same.

5) Cf. Remark 1, § 3.

$$|M_k| \exp(a\delta P_{n_{k-1}}) \leqq \int_0^1 \exp(aS^*_{n_k}(x))dx \leqq 32 \exp\left(\frac{1}{2}a^2 B_{n_k}\right).$$

Putting $a = \delta P_{n_{k-1}}/B_{n_k}$ we deduce easily that

$$(9) \qquad |M_k| \leqq 32 \exp\left(-\frac{1}{2}\delta^2 P^2_{n_{k-1}}/B_{n_k}\right) \quad (k = 2, 3, \ldots).$$

On the other hand, by (6) we have

$$(10) \qquad B_{n_k}/P^2_{n_{k-1}} = (p_1^2 + \cdots + p^2_{n_k})/P^2_{n_{k-1}} \leqq p_{n_k}P_{n_k}/(P_{n_k}/2)^2 = 4p_{n_k}/P_{n_k},$$

and from (4) we have

$$(11) \qquad p_{n_k}/P_{n_k} \leqq \frac{\delta^2}{16}(1/\log\log P_{n_k})$$

for large k, From (5) and (6) we obtain that

$$(12) \qquad P_{n_k} = P_{n_k+1} - p_{n_k+1} \geqq \frac{2}{3}P_{n_k+1} > \frac{4}{3}P_{n_{k-1}}$$

$$> \cdots > (4/3)^{k-1}P_{n_1} > (4/3)^{k-1} \qquad (k = 1, 2, \ldots).$$

Combining (10), (11), (12) and (9) we deduce easily that

$$|M_k| \leqq 32 \exp(-2\log\log(4/3)^{k-1}) = 32/[(k-1)\log(4/3)]^2$$

for large k, and the convergance of (7) is proved, q.e.d.

§ 3. REMARK 1. The inequality (8) is essentially included in [3], but for the sake of completeness we shall prove it here. From the inequality ([3], Lem. 2)

$$\int_0^1 \exp(aS^*_n(x))dx \leqq 16 \int_0^1 \exp(a|S_n(x)|)dx \qquad (a > 0)$$

and the Khintchine inequality (see for example, [2], proof of [456] p. 131)

$$\int_0^1 |S_n(x)|^{2p}dx \leqq \frac{(2p)!}{p!\,2^p}B^p_n \qquad (p = 1, 2, \ldots),$$

we deduce easily that

$$\int_0^1 \exp(aS^*_n(x))\,dx \leqq 32\int_0^1 \cosh(a|S_n(x)|)dx = 32\sum_{p=0}^{\infty}\frac{1}{(2p)!}\int_0^1 a^{2p}|S_n(x)|^{2p}dx$$

$$\leqq 32\sum_{p=0}^{\infty}\frac{1}{(2p)!}\frac{a^{2p}(2p)!}{p!\,2^p}B^p_n = 32\sum_{p=0}^{\infty}\frac{1}{p!}\left(\frac{1}{2}a^2B_n\right)^p = 32\exp\left(\frac{1}{2}a^2B_n\right).$$

REMARK 2. As we can see in the proof of Theorem, the condition (4) may be replaced by the condition

$$B_n/P_n^2 = o(1/\log\log P_n), \tag{13}$$

but these two conditions (4) and (13) are equivalent to each other. In fact, (4) implies (13), for $B_n/P_n^2 \leqq p_nP_n/P_n^2 = p_n/P_n$, and we shall show that (13) involves (4). For sufficiently large n we have $p_n/P_n < 1/4$, and as in the proof of Theorem we can find an integer $m = m(n) > n$ such that $P_n < P_m \leqq 2P_n$ and $P_{m+1} > 2P_n$. From these inequalities we have

$$\frac{p_n}{P_n} = \frac{p_nP_m}{P_nP_m} \leqq \frac{p_nP_m}{P_nP_m} + \frac{p_{n+1}^2 + \cdots + p_m^2}{P_nP_m} \leqq \frac{p_nP_{m+1}}{P_n2P_m} + \frac{2B_m}{P_m^2}$$

and $P_{m+1}/P_m = P_{m+1}/(P_{m+1}-p_{m+1}) = 1/(1-p_{m+1}/P_{m+1}) < 4/3$, hence we have $p_n/P_n < (2/3)p_n/P_n + 2B_m/P_m^2$, that is,

$$p_n/P_n \leqq 6B_m/P_m^2 = o(1/\log\log P_m) = o(1/\log\log P_n).$$

REMARK 3. We shall add a simple proof of the Hill theorem (iii). If $\varphi_n(x)$ converges in a set of positive measure, it does almost everywhere by (i); and if its limit is not essentially constant, we can find two disjoint sets P and Q of positive measure such that every limit of $\varphi_n(x)$ for $x \in P$ is greater than any limit for $x \in Q$. However, by the Steinhaus theorem ([5]), we can obtain two points $p \in P$ and $q \in Q$ whose distance is a dyadic rational; and for such points the limits of $\varphi_n(x)$ are clearly equal, which contradicts the above fact. Hence $\varphi_n(x)$ converges to a constant, c say, almost everywhere. And we can find a point t such that $\varphi_n(x)$ converges to c for both $x = t$ and $x = 1-t$; then the evident relation $\varphi_n(t) = -\varphi_n(1-t)$ shows that $c = -c$, that is, $c = 0$. Finally, we have, for almost all x, $|p_n/P_n| = |p_nr_n(x)/P_n| = |(S_n(x)-S_{n-1}(x))/P_n| \leqq |S_n(x)/P_n| + |S_{n-1}(x)/P_n| \leqq |\varphi_n(x)| + |\varphi_{n-1}(x)|$, which tends to zero, that is, $p_n/P_n \to 0$ as $n \to \infty$, q.e.d.

REMARK 4. If $\{p_n/P_n\}$ is a non-increasing sequence, the condition (2) implies $p_n/P_n = o(1/\log P_n)$ and a fortiori our condition (4). In fact, for $\varepsilon > 0$, we have for sufficiently large m and for any $n > m$,

$$\varepsilon > \sum_{k=m}^{n} (p_k/P_k)^2 \geqq (p_n/P_n) \sum_{k=m}^{n} (p_k/P_k) \sim (p_n/P_n) \log P_n$$

as $n \to \infty$, in virtue of the well known Cesàro theorem. Hence in this case the Hill theorem (ii) is a consequence of ours.

References.

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