28. On the Simple Extension of a Space with Respect to a Uniformity. II.

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The present note is a continuation of our previous study concerning the simple extension of a space with respect to a uniformity[1]. As an application we deduce Shanin's theory on the bicompact extensions of topological spaces[2]. We use the same terminologies and notations as in the first note which will be cited with I.

§ 1. **A characterization of the simple extension.** Let R^* be the simple extension of a *space* R with respect to a *uniformity* $\{\mathfrak{U}_\alpha;\ \alpha \in \Omega\}$[3]. Then we have

Lemma 1. *For an open set G of R it holds that $G^* = R^* - \overline{R-G}$, where the bar indicates the closure operation in R^*.*

Proof. Since $(R-G)\cdot G^* = 0$ by I, Lemma 5, we have $R-G \subset R^*-G^*$ and hence $\overline{R-G} \subset R^*-G^*$. On the other hand, if $x \in R^*-G^*$, then, for any open set H of R such that $x \in H^*$, we have $H^*(R^*-G^*) \neq 0$, and hence $H^*(R-G) \neq 0$; this shows that $R^*-G^* \subset \overline{R-G}$.

Theorem 1. *The simple extension R^* of a space R with respect to a uniformity $\{\mathfrak{U}_\alpha;\ \alpha \in \Omega\}$ is characterized as a space S with the following properties (i.e. such a space S is mapped on R^* by a homeomorphism which leaves each point of R invariant):*

(1) *R is a subspace of S.*

(2) *$\{S-\overline{R-G};\ G$ open in $R\}$ is a basis of open sets of S.*

(3) *Each point of $S-R$ is closed.*

(4) *$\mathfrak{V}_\alpha = \{S-\overline{R-U};\ U \in \mathfrak{U}_\alpha\}$ is an open covering of S.*

(5) *$\{S(x, \mathfrak{V}_\alpha);\ \alpha \in \Omega\}$ is a basis of neighbourhoods at the point x of $S-R$.*

(6) *For any point x of $S-R$ there exists a vanishing Cauchy family $\{X_\lambda\}$ of R (with respect to $\{\mathfrak{U}_\alpha\}$) such that $x = \Pi\bar{X}_\lambda$ in S, and*

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1) K. Morita: On the simple extension of a space with respect to a uniformity, I, the Proc. **27**, No. 2 (1951).

2) N. A. Shanin: Doklady URSS, **38** (1943), pp. 3-6; pp. 110-113; pp. 154-156. These papers are not yet accessible to us; we knew the results only by Math. Reviews.

3) Cf. I, §§ 1 and 3. It is to be noted that a space means here a neighbourhood space such that the family of all open sets containing a point p forms a basis of neighbourhoods of p.

for any vanishing Cauchy family $\{P_\mu\}$ *of* R *we have* $\Pi\bar{P}_\mu \neq 0$ *in* S. *Here the bar indicates the closure operation in* S.

Proof. By virtue of Lemma 1 and I, Lemmas 8–11 we see that R^* has these properties (1)–(6). Let S be another space with the properties (1)–(6). For a vanishing Cauchy family $\{X_\lambda\}$ of the class x which is a point of R^*-R we have $\Pi\bar{X}_\lambda \neq 0$ in S by (6). If $y, z \in \Pi\bar{X}_\lambda$, then $y \in \prod_\alpha S(z, \mathfrak{V}_\alpha)\cdot(S-R)$, since $S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$ implies $X_\lambda \subset S(X_\lambda, \mathfrak{V}_\beta) \subset S-\overline{R-U_\alpha}$ where $U_\alpha \in \mathfrak{U}_\alpha$; and therefore we have $y = z$ by (3) and (5). If $\{Y_\mu\}$ is another Cauchy family of the class x, then for any $\beta \in \Omega$ and $X_\lambda \in \{X_\lambda\}$ there exist $Y_\mu \in \{Y_\mu\}$ and $\gamma \in \Omega$ such that $S(Y_\mu, \mathfrak{U}_\gamma) \subset S(X_\lambda, \mathfrak{U}_\beta)$. Hence, if $S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$, we have $\bar{Y}_\mu \subset S(Y_\mu, \mathfrak{V}_\gamma) \subset S-\overline{R-U_\alpha}$ and consequently $\Pi\bar{X}_\lambda = \Pi\bar{Y}_\mu$. Therefore we can define a mapping f of R^* into S by putting $f(x) = x$ for $x \in R$ and $f(x) = \Pi\bar{X}_\lambda$ for $x \in R^*-R$. By (6) we see that f maps R^* on the whole of S and $f(R^*-R) = S-R$. Let G be an open set of R and x any point of G^*-G. Then for a Cauchy family $\{X_\lambda\}$ of the class x there exist $X_\lambda \in \{X_\lambda\}$ and $\alpha \in \Omega$ such that $S(X_\lambda, \mathfrak{U}_\alpha) \subset G$; the latter relation implies $f(x) \in \bar{X}_\lambda \subset S(X_\lambda, \mathfrak{V}_\alpha) \subset S-\overline{R-G}$. Conversely, if $f(x) = \Pi\bar{X}_\lambda \in (S-\overline{R-G})-R$, then by (5) there exists $\alpha \in \Omega$ such that $S(f(x), \mathfrak{V}_\alpha) \subset S-\overline{R-G}$. Suppose that $S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$, $U_\alpha \in \mathfrak{U}_\alpha$. Then $S-\overline{R-U_\alpha} \subset S(f(x), \mathfrak{V}_\alpha) \subset S-\overline{R-G}$ and hence $U_\alpha \subset G$. This shows that $S(X_\lambda, \mathfrak{U}_\beta) \subset G$, that is, $x \in G^*$. Thus f is a topological mapping of R^* onto S.

§ 2. **A theorem concerning R*.** In this section we shall denote by $\bar{A}$ the closure of A in the space R^*, where R^* is the simple extension of a space R with respect to a uniformity $\{\mathfrak{U}_\alpha;\ \alpha \in \Omega\}$.

Lemma 2. *For a closed set* F *of* R *it holds that*

$$\bar{F} = F + [\prod_\alpha S(F, \mathfrak{U}_\alpha^*)]\cdot(R^*-R).$$

Proof. Since it is clear that $\bar{F}-F \subset R^*-R$, we have only to prove that a point x of R^*-R belongs to $\bar{F}-F$ if and only if $x \in \prod_\alpha S(F, \mathfrak{U}_\alpha^*)$. This follows immediately from the fact that $\{S(x, \mathfrak{U}_\alpha^*)\,;\ \alpha \in \Omega\}$ is a basis of neighbourhoods of x (cf. I, Lemma 9).

Now we can establish the following theorem, which is of importance in applications.

Theorem 2. *Let* G_λ, $\lambda \in \Lambda$ *be open sets of* R. *Suppose that for any* $\alpha \in \Omega$ *there exists some* $\beta \in \Omega$ *such that* $\mathfrak{U}_\beta$ *is a refinement of the covering* $\mathfrak{W}_\alpha$, *where* $\mathfrak{W}_\alpha = \{G_\lambda\,;\ \lambda \in \Lambda\} + \{U\,;\ U(R-\sum G_\lambda) \neq 0,\ U \in \mathfrak{U}_\alpha\}$. *Then the relation*

$$(\sum G_\lambda)^* = \sum G_\lambda^*$$

holds in the space R^*.

Proof. Let us put $L=S(R-\sum G_\lambda, \mathfrak{U}_\alpha^*)$. From the hypothesis of the theorem it follows that $\sum G_\lambda^*+L=R^*$, and hence we have $R^*-\sum G_\lambda^* \subset \prod_\alpha S(R-\sum G_\lambda, \mathfrak{U}_\alpha^*)$. Therefore we have, by Lemma 2, $(R^*-\sum G_\lambda^*)(R^*-R) \subset \overline{R-\sum G_\lambda}-(R-\sum G_\lambda)$, and consequently $R^*-\sum G_\lambda^* \subset \overline{R-\sum G_\lambda}$, since $(R^*-\sum G_\lambda^*)\cdot R = R-\sum G_\lambda$. Thus we have, by Lemma 1, $(\sum G_\lambda)^* \subset \sum G_\lambda^*$. Since the converse relation $\sum G_\lambda^* \subset (\sum G_\lambda)^*$ holds clearly, the proof of Theorem 2 is completed.

Remark. In case R^* is bicompact and $\{\mathfrak{U}_\alpha\}$ is a T-uniformity agreeing with the topology, the converse of Theorem 2 holds.

§ 3. Uniformly continuous mappings.

Theorem 3. *Let f be a uniformly continuous mapping of a space R with a uniformity $\{\mathfrak{U}_\lambda\}$ into a T_1-space S with a regular uniformity $\{\mathfrak{B}_\alpha\}$ agreeing with the topology. Then f can be extended to a uniformly continuous mapping f^* of R^* into S^*. Here R^* or S^* is the simple extension of R or S with respect to the uniformity $\{\mathfrak{U}_\lambda\}$ or $\{\mathfrak{B}_\alpha\}$ and a uniformly continuous mapping is defined as such a mapping φ that for any $\mathfrak{B}_\alpha \in \{\mathfrak{B}_\alpha\}$ there exists a refinement $\mathfrak{U}_\lambda \in \{\mathfrak{U}_\lambda\}$ of the covering $\varphi^{-1}(\mathfrak{B}_\alpha)$*[4].

Proof. Let x be a point of R^*-R and $\{X_\sigma\}$ a vanishing Cauchy family of the class x. Then $\{f(X_\sigma)\}$ is clearly a Cauchy family of S with respect to the uniformity $\{\mathfrak{B}_\alpha\}$ and if $\{X_\sigma\}\sim\{Y_\tau\}$, then $\{f(X_\sigma)\}\sim\{f(Y_\tau)\}$. Hence if we put

$$f^*(x) = \prod \overline{f(X_\sigma)}, \quad \text{for } x \in R^*-R,$$
$$f^*(x) = f(x), \quad \text{for } x \in R,$$

where the bar indicates the closure operation in S^*, f^* defines a one-valued mapping of R^* into S^*.

Let $\mathfrak{B}_\alpha$ be any covering of $\{\mathfrak{B}_\alpha\}$. Then if we take a covering $\mathfrak{B}_{\lambda(\alpha)}$ with the property mentioned in the condition (C) of the first note I, § 1 and determine a covering $\mathfrak{U}_\mu \in \{\mathfrak{U}_\lambda\}$ such that $\mathfrak{U}_\mu$ is a refinement of $f^{-1}(\mathfrak{B}_{\lambda(\alpha)})$, we can easily verify that $\mathfrak{U}_\mu^*$ is a refinement of $(f^*)^{-1}(\mathfrak{B}_\alpha^*)$.

§ 4. Shanin's theory. Let $\mathfrak{G}$ be a basis of open sets of a space R such that $\mathfrak{G}$ contains R. Throughout this section we denote by $\{\mathfrak{U}_\alpha\}$ the family of all the finite open coverings $\mathfrak{U}_\alpha$ which consist of a finite number of sets belonging to $\mathfrak{G}$; $\{\mathfrak{U}_\alpha\}$ is not empty, since $\{R\}$ can be regarded as such a covering.

Theorem 4. *The simple extension R^* of R with respect to the uniformity $\{\mathfrak{U}_\alpha\}$ defined above is bicompact.*

Proof. As is shown in the remark at the end of § 3 in I, $\mathfrak{G}^*=\{G^*; G \in \mathfrak{G}\}$ is a basis of open sets of R^*. Hence we have

4) $\varphi^{-1}(\mathfrak{B}_\alpha)$ means a covering of R which consists of $\varphi^{-1}(V)$, $V \in \mathfrak{B}_\alpha$.

only to prove that any open covering $\mathfrak{M}$ of R^* consisting of sets $G_\lambda^*(\lambda \in \wedge)$ of $\mathfrak{G}^*$ has a finite subcovering. We put $\mathfrak{F} = \{F; R-F \in \mathfrak{G}\}$ and denote by $\Delta\mathfrak{F}$ the family of all the finite intersections of sets of $\mathfrak{F}$. Suppose that $\mathfrak{M} = \{G_\lambda^*; \lambda \in \wedge\}$ has no finite subcovering. Then $\{R-G_\lambda; \lambda \in \wedge\}$ has the finite intersection property, and there exists a maximal family $\{X_\gamma\}$ of sets of $\Delta\mathfrak{F}$ which contains $\{R-G_\lambda; \lambda \in \wedge\}$ and has the finite intersection property. Let $\mathfrak{U}_\alpha=\{G_1, \dots, G_m\}$ be any covering of $\{\mathfrak{U}_\alpha\}$. Then by the maximality of $\{X_\gamma\}$ we have $X_\gamma \cdot (R-G_j) = 0$ for some $X_\gamma \in \{X_\gamma\}$ and some j. Since $X_\gamma \in \Delta\mathfrak{F}$, X_γ is expressed as $\prod_{i=1}^{n}(R-H_i)$ with $H_i \in \mathfrak{G}\,(i = 1, 2, \dots, n)$. Then $\{G_j, H_1, \dots, H_n\}$ is a covering of R and hence it is equal to some $\mathfrak{U}_\beta \in \{\mathfrak{U}_\alpha\}$, and we have $S(X_\gamma, \mathfrak{U}_\beta) \subset G_j$. This shows that $\{X_\gamma\}$ is a Cauchy family in R with respect to $\{\mathfrak{U}_\alpha\}$. Hence we have $\prod \bar{X}_\gamma \neq 0$ in R^* and therefore $\overline{\prod R-G_\lambda} \neq 0$. The last relation contradicts the assumption that $\{G_\lambda^*\}$ is a covering of R^*.

Next we shall show that the simple extension R^* in the present case is characterized as a *bicompact* space S with the properties:

1) R is a subspace of S.

2) $\{S-\overline{R-G}; G \in \mathfrak{G}\}$ is a basis of open sets of S.

3) Each point of $S-R$ is closed.

4) For any finite number of sets $G_1, \dots, G_m$ of $\mathfrak{G}$, $G_1 + \dots + G_m = R$ implies $\sum_{i=1}^{m}(S-\overline{R-G_i}) = S$.

Here the bar indicates the closure operation in S.

In view of Theorem 1 it is sufficient to prove that the conditions (5) and (6) in Theorem 1 follow from 1)-4) and the bicompactness of S, since it is clear that (1)-(4) are implied by 1)-4).

Let $x \in S-R$. Then there exist $F_\lambda(\lambda \in \wedge)$ such that $x = \prod \bar{F}_\lambda$ and $R-F_\lambda \in \mathfrak{G}$, since x is closed by 3). If $x \in S-\overline{R-U_\alpha}$ for some $U_\alpha \in \mathfrak{U}_\alpha$ there exist a finite number of sets $F_{\lambda i}$, $\lambda_i \in \wedge$, $i = 1, 2, \dots, n$ such that $\prod \bar{F}_{\lambda i} \subset S-\overline{R-U_\alpha}$, since S is bicompact. This shows that $S(\prod F_{\lambda i}, \mathfrak{U}_\beta) \subset U_\alpha$, where $\mathfrak{U}_\beta = \{U_\alpha, R-F_{\lambda_1}, \dots, R-F_{\lambda n}\}$. Therefore $\{\prod_{i=1}^{m} F_{\nu i}; \nu_i \in \wedge, m = 1, 2, \dots\}$ is a vanishing Cauchy family in R with respect to $\{\mathfrak{U}_\alpha\}$, and we have $S(x, \mathfrak{V}_\beta) \subset S-\overline{R-U_\alpha}$. Combining this result with the bicompactness of S we see that (5) and (6) hold. Thus we obtain

Theorem 5. *Let $\mathfrak{G}$ be a basis of open sets of a space R such that $\mathfrak{G}$ contains R as an element. Then there exists a bicompact space S with the properties 1)–4). Moreover such a space S is essentially unique in the sense that any space S with these properties is mapped on the simple extension R^* of R (with respect to $\{\mathfrak{U}_\alpha\}$ defined above) by a homeomorphism which fixes each point of R.*

The space S in Theorem 5 is called *the bicompact extension of R with respect to an open basis $\mathfrak{G}$.*

Supplement to Theorem 5. *For the space S in Theorem 5 we have*

$$5)\qquad S-\overline{R-\sum_{i=1}^{m} G_i} = \sum_{i=1}^{m}(S-\overline{R-G_i}),$$

where G_i $(i=1, 2,\dots, m)$ are any finite number of sets of $\mathfrak{G}$[5].

Proof. The relation follows immediately from Theorems 2 and 5.

Now let us assume that R is a T-space and an open basis $\mathfrak{G}$ satisfies the further condition:

$$\beta)\qquad G, H \in \mathfrak{G} \text{ implies } G\cdot H \in \mathfrak{G}.$$

Then the uniformity $\{\mathfrak{U}_a\}$, which consists of all the finite open coverings with sets of $\mathfrak{G}$ as elements, is a T-uniformity. Hence the simple extension R^* of R with respect to this uniformity is a T-space by I, Theorem 5. If we put $\mathfrak{F}=\{F;\ R-F\in\mathfrak{G}\}$, the condition $\beta)$ is transformed into the following condition:

$$\beta)'\qquad F, K\in\mathfrak{F} \text{ implies } F+K\in\mathfrak{F}$$

and Theorem 5 may be stated as follows.

Theorem 6. *Let $\mathfrak{F}$ be a basis of closed sets of a T-space R such that the empty set belongs to $\mathfrak{F}$ and $\mathfrak{F}$ satisfies the condition $\beta)'$. Then there exists a bicompact T-space S with the following properties:*

1)′ *R is a subspace of S.*

2)′ *$\{\bar{F};\ F\in\mathfrak{F}\}$ is a basis of closed sets of S.*

3)′ *Each point of $S-R$ is closed.*

4)′ *For any finite number of sets $F_1,\dots, F_m$ of $\mathfrak{F}$, $F_1F_2\dots F_m=0$ implies $\bar{F}_1\cdot\bar{F}_2\dots\bar{F}_m=0$.*

Here the bar indicates the closure operation in S. Moreover such a space S is essentially unique.

Supplement to Theorem 5 we may replace the condition 4)′ by the stronger condition:

5)′ $\overline{F_1\cdot F_2\dots F_m}=\bar{F}_1\cdot\bar{F}_2\dots\bar{F}_m$ for any finite number of sets $F_1, F_2,\dots, F_m$ of $\mathfrak{F}$.

Theorem 6 with the conditions 1)′, 2)′, 3)′, 5)′ is established by N. A. Shanin[6]. We call S in Theorem 6 *the bicompact extension of R with respect to a closed basis $\mathfrak{F}$.* N. A. Shanin called S the $(\omega, \mathfrak{F})$-extension.

5) The condition 4) is implied by 5).

6) Loc. cit., 2). Shanin's proof is not known to us. We remark here that a generalization of Wallman's famous procedure gives a direct proof of this theorem.

The condition of Shanin:

$\alpha)'$ For any point $x \in R$ and $F \in \mathfrak{F}$ such that $x \in R-F$ there exist $F_i \in \mathfrak{F}$, $i = 1, 2, \ldots, m$ with the properties: $x \in \prod_{i=1}^{m} F_i$ and $\left(\prod_{i=1}^{m} F_i\right) \cdot F = 0$,
is stated in terms of $\mathfrak{G}$ as follows.

$\alpha)$ For any $x \in R$ and $G \in \mathfrak{G}$ such that $x \in G$ there exist $G_i \in \mathfrak{G}$, $i = 1, 2, \ldots, m$ with the properties: $x \in R - \sum_{i=1}^{m} G_i$, $G + \sum_{i=1}^{m} G_i = R$.
This condition is equivalent to the condition that the *uniformity* $\{\mathfrak{U}_\alpha\}$[7] *agrees with the topology of* R. Another condition of Shanin:

$\delta)'$ For $F_1, F_2 \in \gamma\mathfrak{F}$ such that $F_1 \cdot F_2 = 0$ there exist $K_1, K_2 \in \gamma\mathfrak{F}$ such that $F_i K_i = 0$, $i = 1, 2$ and $K_1 + K_2 = R$, is expressed in terms of $\mathfrak{G}$ as follows.

$\delta)$ For $G_1, G_2 \in \gamma\mathfrak{G}$ such that $G_1 + G_2 = R$ there exist $H_1, H_2 \in \gamma\mathfrak{G}$ such that $G_i + H_i = R$, $i = 1, 2$ and $H_1 \cdot H_2 = 0$[8].
We shall prove that *the condition* $\delta)$ *is equivalent to the condition that* $\{\mathfrak{U}_\alpha\}$ *is a completely regular uniformity* (in case $\beta)$ holds).

Suppose that $\delta)$ holds. For any covering $\mathfrak{U}_\alpha = \{G_1, \ldots, G_m\} \in \{\mathfrak{U}_\alpha\}$ we can determine a finite number of sets L_ν, $H_\nu (\nu = 1, 2, \ldots, i)$ of $\gamma\mathfrak{G}$ such that

$$L_\nu \subset R - H_\nu \subset G_\nu, \ \nu = 1, 2, \ldots, i,$$

$$L_1 + \cdots + L_i + G_{i+1} + \cdots + G_m = R,$$

by applying the condition $\delta)$ successively to the case $i = 1, 2, \ldots, m$. We have then $G_i + H_i = R$, $i = 1, 2, \ldots, m$ and $H_1 \cdot H_2 \ldots H_m = 0$. If we express H_i as $H_{i1} + H_{i2} + \cdots + H_{ir_i}$ with $H_{ij} \in \mathfrak{G}$, and construct the intersection $\mathfrak{M}$ of the coverings $\{G_i, H_{i1}, H_{i2}, \ldots, H_{ir_i}\}$ $(i = 1, 2, \ldots, m)$, it is easily seen that $\mathfrak{M}$ is a Δ-refinement[9] of $\mathfrak{U}_\alpha$ and $\mathfrak{M}$ is equal to some $\mathfrak{U}_\gamma \in \{\mathfrak{U}_\alpha\}$. Therefore the uniformity $\{\mathfrak{U}_\alpha\}$ is completely regular.

Conversely, if $\{\mathfrak{U}_\alpha\}$ is completely regular, the condition $\delta)$ holds, because for $G_1, G_2 \in \gamma\mathfrak{G}$ such that $G_1 + G_2 = R$ there exists a Δ-refinement $\mathfrak{U}_\gamma$ of $\mathfrak{M}$, where $G_i = G_{i1} + \cdots + G_{ir_i}$, $i = 1, 2$ with $G_{ij} \in \mathfrak{G}$ and $\mathfrak{M} = \{G_{ij};\ j = 1, 2, \ldots, r_i, i = 1, 2\}$, and hence the sets $H_i = \prod_{j=1}^{r_i} S(R - G_{ij}, \mathfrak{U}_\gamma)$, $i = 1, 2$ satisfy the condition $\delta)$.

Summarizing these results we obtain the following theorem of Shanin by virtue of I, Theorems 6 and 7.

7) Cf. The biginning of this section.

8) Here $\gamma\mathfrak{G}$ means the family of all the finite sums of sets of $\mathfrak{G}$. Cf. loc. cit., 2).

9) Cf. J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940, Chap. V.

Theorem 7. *The bicompact extension of a T-space R with respect to an open basis* $\mathfrak{G}$ *(a closed basis* $\mathfrak{F}$*) is a* T_1*-space or a completely regular T-space according as R is a* T_1*-space and* $\mathfrak{G}$ $(\mathfrak{F})$ *satisfies the conditions* $\alpha)$, $\beta)$ $(\alpha)'$, $\beta)')$ *or R is a completely regular T-space and* $\mathfrak{G}$ $(\mathfrak{F})$ *satisfies* $\alpha)$, $\beta)$, $\delta)$ $(\alpha)'$, $\beta)'$, $\delta)')$.

We can also establish the following theorem.

Theorem 8. *Let* $\mathfrak{G}$ *and* $\mathfrak{H}$ *be two bases of open sets of a space R such that* $\mathfrak{H}$ *as well as* $\mathfrak{G}$ *contains R as an element. In order that the bicompact extension of R with respect to* $\mathfrak{G}$ *coincides with the bicompact extension of R with respect to* $\mathfrak{H}$ *it is necessary and sufficient that for any finite number of sets* $G_1, \dots, G_m$ *of* $\mathfrak{G}$ *with the property* $G_1 + \cdots + G_m = R$ *there exist sets* $H_1, \dots, H_n$ *of* $\mathfrak{H}$ *such that* $H_1 + \cdots + H_n = R$ *and each* H_i *is contained in some* G_j *and conversely for any finite number of sets* $H_1, \dots, H_n$ *of* $\mathfrak{H}$ *with* $H_1 + \cdots + H_n = R$ *there exist sets* $G_1, \dots, G_m$ *of* $\mathfrak{G}$ *such that* $G_1 + \cdots + G_m = R$ *and each* G_j *is contained in some* H_i.

The condition of Theorem 8 states that the uniformity $\{\mathfrak{U}_\alpha\}$ and $\{\mathfrak{V}_\lambda\}$ are *equivalent* (cf. I, § 1), where $\{\mathfrak{V}_\lambda\}$ means the family of all the finite open coverings composed of sets of $\mathfrak{H}$. Hence Theorem 8 can be proved easily.

It is a simple matter to establish an analogous theorem for the case of bicompact extensions with respect to closed bases, and so it is omitted here.

§ 5. **Wallman's and Čech's bicompactifications.** Let R be a space. We denote by $w(R)$ the simple extension of R with respect to the uniformity consisting of all the finite open coverings. It is clear by Theorems 5 and 7 that $w(R)$ is bicompact and $w(R)$ coincides with Wallman's bicompactification in case R is a T_1-space. In the case when R is completely regular we can construct the simple extension of R with respect to the uniformity which consists of all the finite *normal*[10] coverings. We denote this space by $\beta(R)$. Since this uniformity is a completely regular uniformity agreeing with the topology, $\beta(R)$ is completely regular and bicompact. It is easily seen by Theorem 3 that a bounded continuous real-valued function on R can be extended to a continuous function on $\beta(R)$. Hence $\beta(R)$ coincides with Čech's bicompactification if R is a T_1-space. In case R is a completely regular T-space it is clear that $w(R)$ coincides with $\beta(R)$ if and only if every finite open covering of R is normal. It is proved independently by Tukey

10) Cf. Tukey, loc. cit., p. 46.

and the present author that the latter condition holds if and only if R is normal[11].

In case R is a completely regular T-space we can prove that $\beta(R)$ is defined as the bicompact extension of R with respect to an open basis $\mathfrak{G}$ which consists of all the open sets G such that $G=\{x\,;\,f(x)>0\}$ for a non-negative bounded continuous function $f(x)$ on R[12].

We shall remark finally that any bounded real-valued continuous function on a space R can be extended to a continuous function on $w(R)$; this is an immediate consequence of Theorem 3[13].

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11) Tukey, loc. cit., p. 47; K. Morita, On uniform spaces and the dimension of compact spaces, Proc. Phys-Math. Soc. Japan **23** (1940), p. 969; Star-finite coverings and the star-finite property, Math. Japonicae, **1** (1948), pp. 60–68.

12) We do not know whether Shanin has constructed $\beta(R)$ in this way or not.

13) This fact was pointed out first by J. Nagata for the case that R is a T_1-space. A direct proof is given in K. Morita, Shijo-Danwakai, 2nd ser. No. 15 (1949), p. 547.