27. Theorems on the Convexity of Bounded Functions.

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§ 1. Introduction.

We denote by R_M the family of functions $\{F_{(z)}\}$ which are regular in $|z|<1$ and have the properties

$$|F(z)|\leqq M \ (M\geqq 1),\ F(0)=0,\ F'(0)=1,$$

and by S_M the family of functions $\{F(z)\}$ which belong to R_M and schlicht in $|z|<1$.

Dieudonne[1] has proved that any function $F(z)$ of the class R_M is schlicht in $|z|<M-\sqrt{M^2-1}$ and this circle is transformed into a starshaped region in w-plane by $w=F(z)$ and the number $M-\sqrt{M^2-1}$ cannot be replaced by any greater one, and R. Nevanlinna[2] has proved that, for any function $F(z)$ which is regular, schlicht in $|z|<1$ and has the properties $F(o)=0$, $F'(o)=1$, the "Rundungsschranke" is $2-\sqrt{3}$.

In this paper, we will find the greatest circle in which any function $F(z)$ of the class R_M is convex, and the "Rundungsschranke" of the class S_M. For this purpose we will show some lemmas in § 2 and will treat the problems cited above in § 3 and 4.

§ 2. Lemmas.

Let $F(z)$ be any function of the class R_M, then

Lemma 1

$$M|z|\frac{1-M|z|}{M-|z|}\leqq|F(z)|\leqq M|z|\frac{1+M|z|}{M+|z|}, \qquad |z|<1.$$

Lemma 2 (Simonart)[3]

$$\frac{(M+|F(z)|)(|F(z)|-M|z|^2)}{M|z|(1-|z|^2)}\leqq|F'(z)|\leqq\frac{(M-|F(z)|)(|F(z)|+M|z|^2)}{M|z|(1-|z|^2)}, |z|<1.$$

Lemma 3[4]

Let $F(z)=\sum_{\nu=1}^{\infty}c_\nu z^\nu$ be regular and $|F(z)|<M$ in $|z|<1$, then

$$M-\frac{|c_1|^2}{M}\geqq|c_2|.$$

For the function $F(z)$ which belongs to the class S_M, the function of ζ.

$$\phi(\zeta) = M^2\frac{F\left(\dfrac{-s+z}{1-\bar{z}s}\right)-F(z)}{M^2-\overline{F(z)}F\left(\dfrac{-s+z}{1-\bar{z}s}\right)}, \qquad |z|<1.$$

is regular and schlicht in $|s|<1$ and has the properties

$$|\phi(\zeta)|<M, \quad \phi(o)=0, \quad \phi(z)=-F(z),$$

and we have

$$\phi'(\zeta) = -M^2\frac{M^2-|F(z)|^2}{\left[M^2-\overline{F(z)}F\left(\dfrac{-\zeta+z}{1-\bar{z}\zeta}\right)\right]^2}F'\left(\frac{-\zeta+z}{1-\bar{z}\zeta}\right)\cdot\frac{1-|z|^2}{(1-\bar{z}\zeta)^2}.$$

Therefore

$$\phi'(o) = -M^2\frac{(1-|z|^2)F'(z)}{M^2-|F(z)|^2}, \quad \phi'(z) = -\frac{M^2-|F(z)|^2}{M^2(1-|z|^2)}.$$

Differentiating $\phi'(\zeta)$ and putting $\zeta=O$, we get

$$\phi''(o) = \frac{M^2(1-|z|^2)F'(z)}{M^2-|F(z)|^2}\left[\frac{F''(z)}{F'(z)}-\frac{2\bar{z}}{1-|z|^2}+\frac{2\overline{F(z)}F'(z)}{M^2-|F(z)|^2}\right].$$

As

$$\psi(\zeta) = \frac{M^2\phi(\zeta)}{\phi'(o)[M-\varepsilon\phi(\zeta)]^2}, \qquad |\varepsilon|=1,$$

is regular and schlicht in $|\zeta|<1$ and

$$\psi(o)=0, \quad \psi(z) = -\frac{M^2F(z)}{\varphi'(o)[M+\varepsilon F(z)]^2},$$

$$\psi'(\zeta) = M^2\frac{M+\varepsilon\phi(\zeta)}{[M-\varepsilon\phi(\zeta)]^3}\,\frac{\phi'(\zeta)}{\phi'(o)},$$

we have

$$\psi'(o)=1, \qquad \psi'(z) = M^2\frac{M-\varepsilon F(z)}{[M+\varepsilon F(z)]^3}\cdot\frac{\phi'(z)}{\phi'(o)}.$$

Hence we have

$$Z\frac{\psi'(z)}{\psi(z)} = z\frac{M-\varepsilon F(z)}{M+\varepsilon F(z)}\cdot\frac{M^2-|F(z)|^2}{M^2(1-|z|^2)F(z)}.$$

As $\psi(\zeta)$ $(\psi(o)=0,\ \psi'(o)=1)$ is regular and schlicht in $|\zeta|<1$, we have

$$\frac{1-|\zeta|}{1+|\zeta|} \leqq \left| s\frac{\psi'(\zeta)}{\psi(\zeta)} \right| \leqq \frac{1+|\zeta|}{1-|\zeta|}, \quad |\zeta|<1$$

and

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leqq |\psi(\zeta)| \leqq \frac{|\zeta|}{(1-|\zeta|)^2}, \quad |\zeta|<1.$$

Putting $\zeta = z$ in these two inequalities and taking $\varepsilon = \frac{|F(z)|}{F(z)}$ or $\varepsilon = -\frac{|F(z)|}{F(z)}$, as $\varepsilon(|\varepsilon| = 1)$ is arbitrary, we can obtain the following lemmas.

Lemma 4

$$\left(1+\frac{|F(z)|}{M}\right)^2 \frac{|z|}{(1+|z|)^2} \leqq |F(z)| \leqq \left(1-\frac{|F(z)|}{M}\right)^2 \frac{|z|}{(1-|z|)^2}, \quad |z|<1,$$

Lemma 5

$$\frac{M+|F(z)|}{M-|F(z)|}\cdot\frac{1-|z|}{1+|z|} \leqq \left| z\frac{F'(z)}{F(z)} \right| \leqq \frac{M-|F(z)|}{M+|F(z)|}\cdot\frac{1+|z|}{1-|z|}, \quad |z|<1.$$

From Lemma 4 and 5 we have the "Verzerungssätze", i. e.

Lemma 6

$$M\frac{1+|z|-\sqrt{(1+|z|)^2-4M^{-1}|z|}}{1+|z|+\sqrt{(1+|z|)^2-4M^{-1}|z|}} \leqq |F(z)|$$

$$\leqq M\frac{\sqrt{(1-|z|)^2+4M^{-1}|z|}-(1-|z|)}{\sqrt{(1-|z|)^2+4M^{-1}|z|}+(1-|z|)}, \qquad |z|<1.$$

Lemma 7

$$\frac{[1+M^{-1}|F(z)|]^3}{1-M^{-1}|F(z)|}\cdot\frac{1-|z|}{(1+|z|)^3} \leqq |F'(z)| \leqq \frac{[1-M^{-1}|F(z)|]^3}{1+M^{-1}|F(z)|}\cdot\frac{1+|z|}{(1-|z|)^3}, \quad |z|<1.$$

§ 3. Convexity of the functions of R_M.

Let $F(z)$ be any function of the class R_M, then $\phi(\zeta)$ defined in § 2 is regular in $|\zeta|<1$ and

$$|\phi(\zeta)|<M, \quad \phi(o)=0.$$

Therefore $\phi(\zeta)$ can be expanded in power series,

$$\phi(\zeta) = c_1\zeta + c_2\zeta^2 + \cdots\cdots\cdots, \quad |\zeta|<1,$$

where

$$c_1 = \phi'(o) = \frac{M^2(1-|z|^2)F'(z)}{M^2-|F(z)|^2}$$

$$c_2=\frac{\phi''(o)}{2}=\frac{M^2(1-|z|^2)^2F'(z)}{2(M^2-|F(z)|^2)}\left[\frac{F''(z)}{F'(z)}-\frac{2\bar{z}}{1-|z|^2}+\frac{2\overline{F(z)}F'(z)}{M^2-|F(z)|^2}\right].$$

Hence, by lemma 3, we have

$$M-M^3\frac{(1-|z|^2)^2|F'(z)|^2}{(M^2-|F(z)|^2)^2}\geqq\frac{M^2(1-|z|^2)^2|F'(z)|}{2(M^2-|F(z)|^2)}$$

$$\times\left|\frac{F''(z)}{F'(z)}-\frac{2\bar{z}}{1-|z|^2}+\frac{2\overline{F(z)}F'(z)}{M^2-|F(z)|^2}\right|,$$

whence we have, for $|z|<p_s=M-\sqrt{M^2-1}$

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]\geqq1+\frac{2|z|^2}{1-|z|^2}+\frac{2|zF'(z)|}{M+|F(z)|}-\frac{2|z|(M^2-|F(z)|^2)}{M(1-|z|^2)^2|F'(z)|},$$

for by the theorem due to Dieudonné given in § 1, $F(z)$ is schlicht in $|z|<p_s$ and then $F'(z)\neq0$ there.

The right side in this inequality is not less than

$$1+\frac{2|z|^2}{1-|z|^2}+2\frac{|F(z)|-M|z|^2}{M(1-|z|^2)}-2|z|^2\frac{M-|F(z)|}{(1-|z|^2)(|F(z)|-M|z|^2)},$$

by lemma 2, and this is not less than

$$1+\frac{2r(1-Mr)}{(1-r^2)(M-r)}-\frac{2r(M-2r+Mr^2)}{(1-r^2)(1-2Mr+r^2)}=\frac{M-(4M^2-1)r+3Mr^2-r^3}{(M-r)(1-2Mr+r^2)}$$

by lemma 1, where $r|=z|<p_s=M-\sqrt{M^2-1}$. Therefore, we have, for $|z|=r<p_s$,

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]\geqq\frac{M-(4M^2-1)r+3Mr^2-r^3}{(M-r)(1-2Mr+r^2)}.$$

The equation

$$f(r)\equiv M-(4M^2-1)r+3Mr^2-r^3=0$$

has only one such real root p_c that $o<p_c\leqq p_s\leqq1$, where equality sign holds only when $M=1$, and $f(r)$ is decreasing function of r for $o\leqq r<1$. So that

$$f(r)>0 \text{ for } o\leqq r<p_c.$$

Thus we have, for $o\leqq r<p_c$,

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]>0.$$

That is to say, the circle $|z|<p_c$ is transformed into a convex region by $w=F(z)$. And we have, for the function

$$F(z)=Mz\frac{1-Mz}{M-z} \qquad (*)$$

which belongs to R_M,

$$1+z\frac{F''(z)}{F'(z)}=\frac{M-(4M^2-1)z+3Mz^2-z^3}{(M-z)(1-2Mz+z^2)}$$

and

$$1+R\left[p_c\frac{F''(p_c)}{F'(p_c)}\right]=0.$$

Therefore $|z|<p_c$ is the greatest circle for convexity of any function of the class R_M. If we denote by d the distance from the origin to a boundary point of the mapped region of $|z|<p_c$ by $w=F(z)$, then, by lemma 1,

$$Mp_c\frac{1-Mp_c}{M-p_c}\leqq d\leqq Mp_c\frac{1+Mp_c}{M+p_c},$$

and eqality sign holds for the function

$$F(z)=Mz\frac{1-Mz}{M-z}$$

of the class R_M. Hence we have the following

Theorem 1.

Let $F(z)$ be any regular function in $|z|<1$ such that

$$|F(z)|<M,\quad F(o)=0,\quad F'(o)=1,$$

then the circle $|z|<p_c$ is mapped to a convex region in w-plane by $w=F(z)$, where p_c is the positive root of the equation

$$M-(4M^2-1)x+3Mx^2-x^3=0,$$

which is not greater than 1, and this value cannot be replaced by any greater one.

Further the distance d from the origin to a boundary point of the mapped region, satisfies the relation

$$Mp_c\frac{1-Mp_c}{M-p_c}\leqq d\leqq Mp_c\frac{1+Mp_c}{M+p_c},$$

and the equality is attained by the function (*).

§ 4. The "Rundungsschranke" of S_M.

Let $F(z)$ be any function of the class S_M, then we have already shown that, the regular function $\phi(\zeta)$, satisfies

$$|\phi(\zeta)| < M, \quad \phi(o) = 0, \quad \phi(z) = -F(z),$$

and $\psi(z)$ is schlicht in $|\zeta| < 1$ and $\psi(o) = 0$, $\psi'(o) = 1$.
We get, by simple calculations,

$$\psi''(0) = -(1-|z|^2)\left[\frac{F''(z)}{F'(z)} - \frac{2\bar{z}}{1-|z|^2} + \frac{2\overline{F(z)}F'(z)}{M^2-|F(z)|^2} + \frac{4\varepsilon MF'(z)}{M^2-|F(z)|^2}\right].$$

Being $|\psi''(o)| \leqq 4$, we have,

$$\left| z\frac{F''(z)}{F'(z)} - \frac{2|z|^2}{1-|z|^2} + \frac{2z\overline{F(z)}\, F'(z)}{M^2-|F(z)|^2} + \frac{4\varepsilon MzF'(z)}{M^2-|F(z)|^2} \right| \leqq \frac{4|z|}{1-|z|^2}.$$

Putting $\varepsilon = -\dfrac{|zF'(z)|}{zF'(z)}$ and taking the real part of the left side, we have

$$R\left[z\frac{F''(z)}{F'(z)}\right] \geqq \frac{2|z|^2-4|z|}{1-|z|^2} + \frac{2M|zF'(z)|}{M^2-|F(z)|^2} + \frac{2|zF'(z)|}{M+|F(z)|}.$$

Applying lemma 4, 5 and 6, we have

$$1+R\left[z\frac{F''(z)}{F'(z)}\right] \geqq \frac{1}{M(1-r)\sqrt{D}}[M(1-r)^2\sqrt{D} - 2(M-1)r(1+r)].$$

where

$$r = |z|, \text{ and } D = (1+r)^2 - 4M^{-1}r.$$

As the equation

$$M(1-r)^2\sqrt{D} - 2(M-1)r(1+r) = 0$$

can be reduced to

$$t^3 - 2\left(1+\frac{2}{M}\right)t^2 - 4\left(2-\frac{6}{M}+\frac{1}{M^2}\right)t - \frac{8}{M^2} = 0, \quad t = r + \frac{1}{r}.$$

When we denote the left side by $f(t)$, then we know, that $f(t)$ has one real root t_0 lying in $(2, 4)$, and

$$f(t) < 0 \text{ for } 2 \leqq t < t_0, \quad f(t) > 0 \text{ for } t > t_0.$$

Putting $r + \dfrac{1}{r} = t_0$ then

$$r = \frac{1}{2}(t_0 - \sqrt{t_0^2-4}) \equiv p_0 \quad (2-\sqrt{3} < p_0 < 1)$$

and

$$r < p_0 \text{ for } t > t_0, \quad r > p_0 \text{ for } t < t_0, \quad r = p_0 \text{ for } t = t_0.$$

Hence we conclude that $f(r) > 0$ for $r < p_0$, i. e.

$$M(1-r)^2\sqrt{D} - 2(M-1)r(1+r) > 0,$$

and therefore, for $r < p_0$,

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0 .$$

So that the circle $|z| < p_0$ is transformed into a convex region by $w = F(z)$. On the other hand, for the function defined by

$$F(z) = M\frac{z+1-\sqrt{(z+1)^2 - 4M^{-1}z}}{z+1+\sqrt{(z+1)^2-4M^{-1}z}}$$

of the class S_M,

$$1 + z\frac{F''(z)}{F'(z)} = \frac{1}{M(1-z)\sqrt{D}}\left[M(1-z)^2\sqrt{D} - 2(M-1)z(1+z)\right]$$

where $D = (1+z)^2 - \frac{4}{M}z$, and therefore

$$1 + R\left[p_0\frac{F''(p_0)}{F'(p_0)}\right] = 0 .$$

Hence p_0 is the greatest number for convexity. Consequently we have the

Theorem 2.

Denoting by s_M *the class of functions* $\{F(z)\}$ *such as* $F(z)$ *is regular and schlicht, having the properties*

$$|F(z)| < M (M \geqq 1) , \quad F(o) = 0 , \quad F'(o) = 1 ,$$

then number p_0 *is the "Rundungsschranke" of* S_M, *where*

$$p_0 = \frac{1}{2}(t_0 - \sqrt{t_0^2 - 4}) ,$$

and t_0 *is a root which is not less than 2 of the equation*

$$t^3 - 2\left(1+\frac{2}{M}\right)t^2 - 4\left(2 - \frac{6}{M} + \frac{1}{M^2}\right)t - \frac{8}{M^2} = 0 .$$

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