# 40. *The Computation of the Path of a Ray and the Correction of the Aberrations of a Lens System. Part I.*

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## The Computation of the Path of a Ray.

When we design photographic or other complicated lens systems, corrections for the aberrations must be done, and this will only be done satisfactorily after tedious and laborious calculations. Therefore, to simplify the procedure of this calculations is an important problem and if we deal with this problem as below, it will become fairly easy. Theoretically, the choice of parameters may be quite arbitrary, but it will become clear that the most convenient way for our present purpose is the use of $(h, \theta)$ in Fig. 1, for it makes the calculation quite easy. For instance, the parameter $(s, u)$ in Fig. 2, which is commonly used to trace the path of a ray, is unsuited for our problem to correct the aberrations, because it makes the computation exceedingly complicated one. Besides, the acceptance of the parameter $(h, \theta)$ makes the trigonometrical computation themselves somewhat easier than ordinary method with $(s, u)$. For these reasons, the method of tracing using $(h, \theta)$ will be explained at first.

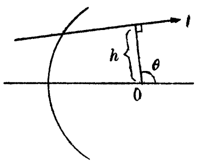



Fig. 1.

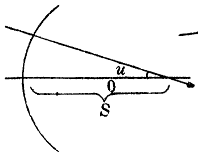



Fig. 2.

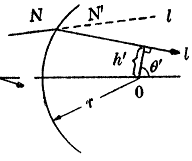



Fig. 3.

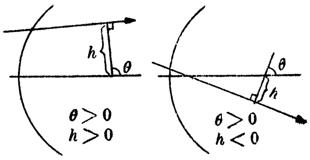



Fig. 4.

In Fig. 1, the symbol $h$ denotes the length of the perpendicular let fall from the centre of the curvature of a refracting surface on the path of a ray and $\theta$ denotes the angle between the perpendicular and the optical axis. Then, $l(h, \theta)$ shows the incident ray and $l'(h', \theta')$ the refracted ray respectively (Fig. 3). The sign of $h$ and $\theta$ is reckoned as shown in Fig. 4. The radius of curvature $r$ of a surface that presents its convex surface towards incident light is regarded as being positive and that of a surface which is concave to the incident light as negative. $N$ and $N'$ mean the refractive indices of the media and $n = N/N$ (Fig. 3). $t$

denotes the distance of two consecutive refracting surfaces along the axis, i.e. , the thickness and is always positive. $C$ denotes the distance of the two consecutive centers of radii of refracting surfaces (Fig. 5).

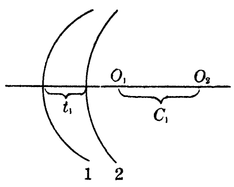



Fig. 5.

$$C_n = r_{n+1} - r_n + t_n.$$

The sign of $C$ must be determined according to the above equation. $s$ denotes the distance $\overline{O_K A}$ (Fig. 6), $O_K$ is the center of the radius of the last surface $K$, and $A$ is the point of intersection of the refracted ray $l'_K(h'_K, \theta'_K)$, i. e., the image point of the incident ray parallel to the first surface, $\bar{s}$ being that of the paraxial ray. The direction of travel of the incident light is reckoned as positive and the sign of $s$ is determined by this convention.

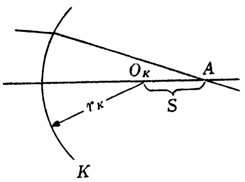



Fig. 6.

(1) Now the following four formulas may be given. If we have determined $(h, \theta)$ of the incident ray, $(h', \theta')$ of the ray after refraction will be found by (1) and (2) ; (3) and (4) enables us to pass to the next surface

$$h' = \frac{h}{n}, \tag{1}$$

$$\theta + \cos^{-1}\frac{h}{r} = \theta' + \cos^{-1}\frac{h'}{r}, \tag{2}$$

$$h_{n+1} = h'_n - C_n \cdot \cos\theta'_n, \tag{3}$$

$$\theta_{n+1} = \theta'_n. \tag{4}$$

Therefore, if we repeat the above calculation for each surface in succession, finally we will obtain the co-ordinates of the emergent ray, viz., $(h'_K, \theta'_K)$.

Then we can reach the final result

$$s = \frac{h'_K}{\cos\theta'_K}.$$

As a special case when the refraction on a plane surface is chosen, putting $r \to \infty$ , we may be able to derive the required formulas quite easily. For the paraxial ray, as $h$ is very small, from (2) and (3) the formulas become as follows:

$$\left.\begin{aligned} \theta' &= \theta - \frac{h}{r}\cdot\left(1-\frac{1}{n}\right), \\ h_{n+1} &= h'_n - C_n\cdot\left(\frac{\pi}{2}-\theta'_n\right), \end{aligned}\right\} \tag{5}$$

Putting $\theta_1 = \pi/2$, repeat the above calculation in succession and we finally obtain $(h'_K, \theta'_K)$. Then, substituting $(h'_K, \theta'_K)$ in (6), we shall have $\bar{s}$ and $f$.

$$\left.\begin{aligned} &= \bar{s}\frac{h'_K}{\dfrac{\pi}{2}-\theta'_K}, \\ f &= \frac{h_1}{\sin u'_K} = \frac{h_1}{\cos\theta'_K} = \frac{h_1}{\dfrac{\pi}{2}-\theta'_K}, \end{aligned}\right\} \tag{6}$$

where $f$: The focal length.

(2) Astigmatism

1. Meridian Ray

The formula for calculating the position of the image-point $P_M$ corresponding to an object-point $P$ on a given chief incident ray $l$ in Fig. 7 may be

$$\frac{1}{P'}+\frac{1}{a'} = n.\left(\frac{1}{P}+\frac{1}{a}\right), \tag{7}$$

where $P = \overline{BP}$, $P' = \overline{BP_m}$, $a = r\cdot\sin\beta$, $a' = r\cdot\sin\beta'$,

$$\beta = \cos^{-1}\frac{h}{r}, \quad \beta' = \cos^{-1}\frac{h'}{r}, \quad \text{(from (2))}.$$

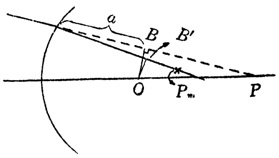
Thus, if the chief incident ray $l$ is given, and if the corresponding chief refracted ray $l'$ has been calculated from (1) and (2), so that the values of $a$ and $a'$ are known, (7) enables us to calculate the value of $P'$ in terms of that of $P$. Then we can pass to the next surface and

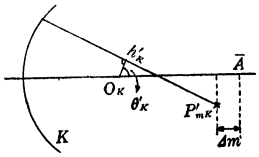
$$P_{n+1} = P'_n - C_n\cdot\sin\theta'_n. \tag{8}$$

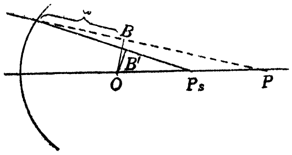
Finally, the value $\varDelta_m$ (Fig. 8) is given as

$$\varDelta_m = h'_K\cdot\cos\theta'_K + P'_K\cdot\sin\theta'_K - \bar{s}.$$

In Fig. 8, $\bar{A}$ is the image-point of the paraxial ray.




$\bar{A}$: focus of the paraxial ray.
$P'_{mk}$: image point.

Fig. 7. Fig. 8. Fig. 9.

2. Sagittal ray

Quite similarly as (7), (8) and (9) (Fig. 9),

$$\frac{1}{s'+a'}-\frac{1}{n(s+a)}=\frac{\sin\beta}{nr}-\frac{\sin\beta'}{r}, \tag{10}$$

where $P$: Object point, $P_S$: Image point,

$$s=\overline{BP}, \qquad s'=\overline{BP_S},$$

and,
$$S_{n+1}=S'_n-C_n\cdot\sin\theta'_n, \tag{11}$$

$$\varDelta S=h'_K\cdot\cos\theta'_K+s'_K\cdot\sin\theta'_K-\bar{s}. \tag{12}$$

(3) Some derivative formulas and relations

1. Graphical representation of the path of a ray is easily performed by repeating the following procedures. First, draw the circle, its centre being coincide to that of the refracting surface and its radius is $h'=h/n$. Next, draw a tangent to this circle from the incident point on the surface, and the tangent shows the path of the refracted ray. Then, this tangent becomes the path of the incident ray to the next surface immediately.

2. Using the parameter $(x, y)$ given below in place of $(h, \theta)$, we may be able to obtain formulas and if we use these formulas, any trigonometric tables to compute the path of a ray are not necessary.

Let us place $x=h/r$, $x'=h'/r$, $y=\cos\theta$ and $y'=\cos\theta'$ in (1), (2), (3) and (4), then we obtain

$$x'=\frac{x}{n}, \tag{13}$$

$$xy-\sqrt{1-x^2}\sqrt{1-y^2}=x'y'-\sqrt{1-x'^2}\sqrt{1-y'^2}, \tag{14}$$

$$x_{n+1}=\frac{r_n}{r_{n+1}}x'_n-C_n y'_n, \tag{15}$$

$$y_{n+1}=y'_n, \tag{16}$$

as modified forms of (1), (2), (3) and (4).

In consequence of such modification as shown below, the calculation of $y$ from (14) becomes much easier.

Put $xy-\sqrt{1-x^2}\sqrt{1-y^2}=Z$, then,

$$xy-\sqrt{(1+xy+x+y)(1+xy-x-y)}=Z\,, \qquad \text{and also,}$$

$$x'y'-\sqrt{1-x'^2}\sqrt{1-y'^2}=Z\,. \qquad \text{Therefore,}$$

$$y'=zx+\sqrt{1-z^2}\sqrt{1-x'^2}=zx'+\sqrt{(1+zx'+z+x')(1+zx'-z-x')}\,.$$

3. Differential formulas to compute the approximate value of a chromatic aberration.

From (1)~(4), we can easily obtain the differential formulas

$$dh'=\frac{dh}{n}-h'\frac{dn}{n}\,, \tag{17}$$

$$d\theta'=d\theta+(\cot\beta'-\cot\beta)\frac{dh}{h}-\cot\beta'\cdot\frac{dn}{n}\,, \tag{18}$$

$$dh_{n+1}=dh'_n+C_n\cdot\sin\theta'_n\cdot d\theta'_n\,, \tag{19}$$

$$d\theta_{n+1}=d\theta'_n\,, \tag{20}$$

where $$dh_1=0\,, \qquad d\theta_1=0\,.$$

For instance, to compute the chromatic aberration of the line $F$ to the line $D$, the only necessary calculations are to repeat (17) ~(20) using the values of $(h,\theta)$, which have been obtained previously when we traced about the line $D$, and the dispersion $(n_F-n_D)/n_0=dn/n$.

Chromatic aberration $s_F-s_D=ds$ is

$$ds=s\left\{\left(\frac{dh_K}{h_K}-\frac{dn_K}{n_K}\right)+\tan\theta'_K\cdot d\theta'_K\right\}. \tag{21}$$

Above calculations (17)~(21) do not involve precise procedures as in the ordinary trigonometric computation, and in general, precise calculations using trigonometric tables may be replaced by that of using slide rule without serious errors.* As for the paraxial ray, (17)~(21) will be much more simplified.

* The error obtained when the author tried about the rim $F$ and $D$ line, (the sample was the achromatic doublet objective, $f=100$ m.m. $F=5$) was about 3.3% compared to that obtained by precise trigonometric tracing.