37. On the Simple Extension of a Space with Respect to a Uniformity. III.

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In the present note we discuss the completion of a space with respect to a uniformity. We make use of the same terminologies and notations as in the previous notes.[1]

§ 1. **The completion for the general case.** Let $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ be a uniformity of a space R. Then the simple extension R^* of R with respect to $\{\mathfrak{U}_\alpha\}$ is complete with respect to the uniformity $\{\mathfrak{U}_\alpha^*\}$, in case $\{\mathfrak{U}_\alpha\}$ is a regular uniformity agreeing with the topology of R.[2] In the general case the simple extension R^*, however, is not always complete with respect to $\{\mathfrak{U}_\alpha^*\}$. We shall treat such a case in the following lines. In this case we construct the simple extension R^{**} of R^* with respect to the uniformity $\{\mathfrak{U}_\alpha^*\}$. Here we shall remark

Lemma 1. *The set of G^{**} for all open sets G of R is a basis of open sets of R^{**}.*

In case R^{**} is not complete we construct further the simple extension of R^{**}, and so on. We carry out our construction by transfinite induction. For the sake of convenience we write $R^{(0)}$, $R^{(1)}$, $R^{(2)}$, ... instead of R, R^*, R^{**}, ... Suppose that $R^{(\nu)}$ (and $G^{(\nu)}$ for open sets G of R) are defined for all ordinals ν less than an ordinal λ, and that $R^{(\nu)}$ are not complete, but with the following properties:

(1) For $0 \leqq \mu < \nu$ we have $R^{(\mu)} \subset R^{(\nu)}$ and $G^{(\nu)} \cdot R^{(\mu)} = G^{(\mu)}$.

(2) $G \subset H$ or $G \cdot H = 0$ implies $G^{(\nu)} \subset H^{(\nu)}$ or $G^{(\nu)} \cdot H^{(\nu)} = 0$.

(3) $\{G^{(\nu)}; G \text{ open in } R\}$ is a basis of open sets of $R^{(\nu)}$.

(4) Each point of $R^{(\nu)} - R$ is closed in $R^{(\nu)}$.

(5) $\mathfrak{U}_\alpha^{(\nu)} = \{U^{(\nu)}; U \in \mathfrak{U}_\alpha\}$ is an open covering of $R^{(\nu)}$.

(6) $\{S(x, \mathfrak{U}_\alpha^{(\nu)}); \alpha \in \Omega\}$ is a basis of neighbourhoods of each point x of $R^{(\nu)} - R$.

Here G, H are open sets of R.

In case λ is not a limit-number, we define $R^{(\lambda)}$ as the simple extension of $R^{(\lambda-1)}$ with respect to the uniformity $\{\mathfrak{U}_\alpha^{(\lambda-1)}; \alpha \in \Omega\}$. Then it is easily seen that $R^{(\lambda)}$ satisfies the conditions (1), (2), (3), (5), (6) for $\nu = \lambda$. If x is a point of $R^{(\lambda)} - R^{(\lambda-1)}$, then x is clearly a closed set of $R^{(\lambda)}$. Let $x \in R^{(\lambda-1)} - R$. Then we have $\bar{x} \cdot R^{(\lambda-1)} = x$.

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1) K. Morita: On the simple extension of a space with respect to a uniformity. I, II. these Proc. **27**, No. 1, 2 (1951). These notes shall be cited with I., II. respectively.

2) Cf. I. § 5.

For $y \in R^{(\lambda)} - R^{(\lambda-1)}$, $x \in S(y, \mathfrak{U}_\alpha^{(\lambda)})$ for any $\alpha \in \Omega$ implies $x \in \bar{y}$, that is, $x = y$. Therefore the condition (4) holds for $\nu = \lambda$.

In case λ is a limit-number, we put

$$R^{(\lambda)} = \sum_{\nu<\lambda} R^{(\nu)}, \qquad G^{(\lambda)} = \sum_{\nu<\lambda} G^{(\nu)}$$

and take $\{G^{(\nu)};\ G \text{ open in } R\}$ as a basis of open sets of $R^{(\lambda)}$. Then it is easily seen that the conditions (1), (2), (3), (5) are satisfied for $\nu = \lambda$. Let $x \in R^{(\lambda)} - R$, and $x \in G^{(\lambda)}$. Then there exists an ordinal ν such that $x \in R^{(\nu)} - R$ and $\nu < \lambda$. By (6) there exists $\alpha \in \Omega$ such that $S(x, \mathfrak{U}_\alpha^{(\nu)}) \subset G^{(\nu)}$, and hence we have $S(x, \mathfrak{U}_\alpha^{(\lambda)}) \subset G^{(\lambda)}$. This shows that the condition (6) holds for $\nu = \lambda$. Next let x be a point of $R^{(\lambda)} - R$ and $y \in \bar{x}$. Then there exists some ordinal ν such that $x, y \in R^{(\nu)}$ and $\nu < \lambda$. Hence we have $y \in \bar{x}$ in the space $R^{(\nu)}$, and consequently we have $y = x$. Therefore the condition (4) is valid also for $\nu = \lambda$.

Thus for any ordinal ν we can define $R^{(\nu)}$ which possesses the properties (1)–(6). Now we have

$$x = [\prod_\alpha S(x, \mathfrak{U}_\alpha^{(\nu)})] \cdot (R^{(\nu)} - R)$$

for any point x of $R^{(\nu)} - R$, by virtue of (4) and (6). Hence for any point x of $R^{(\nu)} - R$ there corresponds a subfamily $\{U;\ U \in \mathfrak{U}_\alpha,\ x \in U^{(\nu)}\}$ of $\mathfrak{U}$ with the finite intersection property, where $\mathfrak{U} = \{U;\ U \in \mathfrak{U}_\alpha,\ \alpha \in \Omega\}$. Therefore, if we denote by $\mathfrak{m}$ the cardinal number of the set $\mathfrak{U}$, it is seen that the cardinal number of $R^{(\nu)} - R$ cannot exceed $2^{\mathfrak{m}}$. Hence for some λ with $|\lambda| \leqq 2^{\mathfrak{m}}$ the space $R^{(\lambda)}$ must be complete with respect to $\{\mathfrak{U}_\alpha^{(\lambda)};\ \alpha \in \Omega\}$. We denote this $R^{(\lambda)}$ by $\tilde{R}$. Here we can easily prove (cf. I, Lemma 1)

Lemma 2. *For an open set G of R we have $G^{(\lambda)} = R - \overline{R - G}$, where the bar indicates the closure operation in $\tilde{R} = R^{(\lambda)}$.*

Therefore we have established the following theorem.

Theorem 1. *Let R be a space with a uniformity $\{\mathfrak{U}_\alpha;\ \alpha \in \Omega\}$ Then there exists a space S with the following properties:*

1) *S contains R as a subspace.*
2) *$\{S - \overline{R - G};\ G$ open in $R\}$ is a basis of open sets of S.*
3) *Each point of $S-R$ is closed.*
4) *$\mathfrak{V}_\alpha = \{S - \overline{R - U};\ U \in \mathfrak{U}_\alpha\}$ is an open covering of S.*
5) *$\{S(x, \mathfrak{V}_\alpha);\ \alpha \in \Omega\}$ is a basis of neighbourhoods of each point x of $S - R$.*
6) *S is complete with respect to the uniformity $\{\mathfrak{V}_\alpha;\ \alpha \in \Omega\}$. Here the bar indicates the closure operation in S.*

Theorem 2. *Any space S, which has the properties* 1)–6) *and is minimal with regard to these properties, is mapped on $\tilde{R}$ by a homeomorphism which leaves each point of R invariant.*

Proof. Let us put $f(x)=x$ for every point x of R. For a point x of $R^{(1)}-R$ there exists a vanishing Cauchy family $\{X_\lambda\}$ in R which belongs to the class x. Then $\{X_\lambda\}$ is also a Cauchy family with respect to $\{\mathfrak{V}_\alpha\}$. Hence $\Pi\bar{X}_\lambda$ is a point of $S-R$. We put $f^{(1)}(x)=\Pi\bar{X}_\lambda$ and $f^{(1)}_{(x)}=f(x)$ for $x\in R$. Then we see easily, as in II, §1, that $f^{(1)}$ maps $R^{(1)}$ onto a subspace of S topologically. By transfinite induction we can construct a homeomorphism $f^{(\lambda)}$ of $R^{(\lambda)}$ into S which is an extension of f. According to the minimal property of S we have $f^{(\lambda)}(\tilde{R})=f^{(\lambda)}(R^{(\lambda)})'=S$.

We call $\tilde{R}$ *the completion of* R *uith respect to the uniformity* $\{\mathfrak{U}_\alpha\}$.

Example. We shall give here a metrizable space with a uniformity agreeing with the topology whose simple extension is not complete.[3] Let

$$R=\{(x,y);\ \ 0\leqq x\leqq 1,\ \ 0<y\leqq 1\}$$

be a subspace of Euclidean plane, and let us put

$$\mathfrak{U}_m=\{V_m(p);\ p\in R\}+\{W_{mj};\ j=1,2,\ldots,m+1\},$$

where $V_m(p)=\{(\xi,\eta);\ |\xi-x|<\frac{1}{2^m},\ |\eta-y|<\frac{1}{2^m}y\}\cdot R$ with

$p=(x,y)$ and

$$W_{mi}=\left\{(x,y);\ \frac{1}{2^i}<x<\frac{1}{2^{i-1}},\ 0<y<\frac{1}{2^m}\right\},\ i=1,2,\ldots,m,$$

$$W_{m,m+1}=\left\{(x,y);\ 0<x<\frac{1}{2^m},\ \ 0<y<\frac{1}{2^m}\right\}.$$

Then $\{\mathfrak{U}_m;\ m=1,2,\ldots\}$ is a T-uniformity of R which agrees with the topology. If we put

$$X^{(i)}_n=\left\{\left(\frac{1}{2^i}+\frac{1}{2^{i+1}},\ \frac{1}{2^k}\right);\ k=n,\ n+1,\ \ldots\right\},$$

$\{X^{(i)}_n;\ n=1,2,\ldots\}$ is a vanishing Cauchy family and determines a point of R^* which will be denoted by ζ_i. It is easily shown that

$$R^*=R+\sum_{i=1}^{\infty}\zeta_i;\qquad V^*_m(p)=V_m(p),$$

$$W^*_{mi}=W_{mi}+\zeta_i,\ i=1,2,\ldots,m;\ \ W^*_{m,m+1}=W_{m,m+1}+\sum_{i=m+1}^{\infty}\zeta_i.$$

In R^*, $\{\sum_{i=n}^{\infty}\zeta_i;\ n=1,2,\ldots\}$ is a vanishing Cauchy family with respect to $\{\mathfrak{U}^*_m\}$. Hence R^* is not complete.

§2. **Regular uniformity.** From Theorem 3 in II we easily obtain the following theorem.

3) Cf. the example at the end of I.

Theorem 3. *Any uniformly continuous mapping f of a space S with uniformity $\{\mathfrak{B}_\lambda\}$ into a T_1-space R with a regular uniformity $\{\mathfrak{U}_\alpha\}$ agreeing with the topology can be extended to a uniformly continuous mapping F of $\tilde{S}$ into $\tilde{R}$, where $\tilde{S}$ and $\tilde{R}$ are the completions of S and R respectively.*

Theorem 4. *Let f be a continuous mapping of a subspace X of a T-space S into a T_1-space R, and let $\{\mathfrak{U}_\alpha;\ \alpha\in\Omega\}$ be a regular T-uniformity of R agreeing with the topology. Then f can be extended to a continuous mapping of X_0 into R^*, where $X\subset X_0\subset S$, $X_0=\prod_{\alpha\in\Omega} H_\alpha\cdot\bar{X}$ with some open sets H_α of S, and R^* means the simple extension of R with respect to $\{\mathfrak{U}_\alpha\}$.*

Proof. Without loss of generality we may assume that $\bar{X}=S$. For an open set G of X we put $\tau(G)=S-\overline{X-G}$. If we put further

$$(7)\quad X_0=\prod_{\alpha\in\Omega} H_\alpha,\qquad H_\alpha=\sum_{U\in\mathfrak{U}_\alpha}\tau[f^{-1}(U)],$$

then H_α are open sets of S and we have $X\subset X_0$. Let x be a point of X_0. Then a family $\{U_\alpha;\ \alpha\in\Omega\}$ such that $x\in\tau[f^{-1}(U_\alpha)]$, $U_\alpha\in\mathfrak{U}_\alpha$ is a Cauchy family in R with respect to $\{\mathfrak{U}_\alpha\}$, according to I, Lemma 16.[4] Another family $\{V_\alpha;\ \alpha\in\Omega\}$ such that $x\in\tau[f^{-1}(V_\alpha)]$, $V_\alpha\in\mathfrak{U}_\alpha$ is equivalent to the above $\{U_\alpha;\ \alpha\in\Omega\}$, since for any α and U_β we have $V_\alpha\subset S(U_\beta,\mathfrak{U}_\alpha)$ (cf. I, Lemma 17). Hence if we put

$$(8)\quad f_0(x)=\prod_{\alpha\in\Omega}\bar{U}_\alpha \qquad\text{in}\quad R^*,$$

f_0 defines a one-valued mapping of X_0 into R^*. f_0 clearly coincides with f for the points of X. If two points x, y of X_0 belong to some $\tau[f^{-1}(U_{\lambda(\alpha)})]$ with $U_{\lambda(\alpha)}\in\mathfrak{U}_{\lambda(\alpha)}$, then $f_0(x)$ and $f_0(y)$ belong to $R^*-\overline{R-U_\alpha}$, where $\mathfrak{U}_{\lambda(\alpha)}$ is a covering with the property mentioned in the condition (C) of I, §1 and $S(U_{\lambda(\alpha)},\mathfrak{U}_\delta)\subset U_\alpha$, $U_\alpha\in\mathfrak{U}_\alpha$. Therefore f_0 is continuous.

Corollary. *A continuous mapping of a subspace X of T-space S into a complete metric space R, can be extended to a continuous mapping of a G_δ-set $X_0\supset X$ into R, where X is assumed to be dense in S.*

Remark. It is well known that the famous theorem of Lavrentieff follows from this corollary.

Now let $\{\mathfrak{U}_\alpha;\ \alpha\in\Omega\}$ be a regular T-uniformity of a T_1-space R agreeing with the topology. If we put $S=w(R)$ (Wallman's bicompactification) and $f(x)=x$ for $x\in R$, and apply Theorem 4 to this case, we see that f can be extended to a continuous mapping φ of H into R^*, where

$$(9)\quad H=\prod_{\alpha\in\Omega} H_\alpha,\quad H_\alpha=\sum_{U\in\mathfrak{U}_\alpha}\tau(U),\quad \tau(U)=w(R)-\overline{R-U},$$

4) For open sets H, K of X, $H\cdot K=0$ if and only if $\tau(H)\cdot\tau(K)=0$ since S is a T-space and $\bar{X}=S$.

the bar indicating the closure operation in $w(R)$. Then φ maps H on the whole of R^*; because for a Cauchy family $\{X_\lambda\}$ we have $\Pi \bar{X}_\lambda \neq 0$ in $w(R)$ and $S(X_\lambda, \mathfrak{U}_\gamma) \subset U_\alpha$ with $U_\alpha \in \mathfrak{U}_\alpha$ implies $\bar{X}_\lambda \subset \tau(U_\alpha)$ and hence we have $\Pi \bar{X}_\lambda \subset H$ and consequently $\{X_\lambda\}$ converges to $\varphi(x)$ in R^* for any point $x \in \Pi \bar{X}_\lambda$. As is shown above, $y \in S(x, \tau(\mathfrak{U}_{\lambda(\alpha)}))$ implies $\varphi(y) \in S(\varphi(x), \mathfrak{U}_\alpha^*)$, where

(10) $\tau(\mathfrak{U}_\alpha) = \{\tau(U); U \in \mathfrak{U}_\alpha\}$.

Let $\varphi(x) = \varphi(y)$ and $x \in \tau(U_\alpha)$, $y \in \tau(V_\alpha)$ with U_α, $V_\alpha \in \mathfrak{U}_\alpha$. Then the Cauchy family $\{U_\alpha; \alpha \in \Omega\}$ is equivalent to $\{V_\alpha; \alpha \in \Omega\}$. Therefore for any α there exist U_β, V_γ such that $U_\beta + V_\gamma \subset W_\alpha$ for some $W_\alpha \in \mathfrak{U}_\alpha$, and so we have $\tau(U_\beta) + \tau(V_\gamma) \subset \tau(W_\alpha)$ that is, $y \in S(x, \tau(\mathfrak{U}_\alpha))$. Thus we have

Theorem 5. *Let R be a T_1-space with a regular T-uniformity $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ agreeing with the topology. Then there exists a continuous mappiny φ of a subspace H of $w(R)$ onto the completion R^* of R with respect to $\{\mathfrak{U}_\alpha\}$ with the following properties:*

1) *$y \in S(x, \tau(\mathfrak{U}_\alpha))$ for every $\alpha \in \Omega$ if and only if $\varphi(x) = \varphi(y)$.*

2) *$\varphi(H-R) = R^*-R$; $\varphi(x) = x$ for $x \in R$.*

Since Čech's bicompactification $\beta(R)$ can be defined as the completion of R with respect to a uniformity consisting of all finite normal coverings, we obtain the following known theorem from Theorem 5.

Theorem 6. *Let R be a completely regular T_1-space. Then there exists a continuous mapping φ of $w(R)$ into $\beta(R)$ such that $\varphi(x) = x$ for $x \in R$ and $\varphi(w(R)-R) = \beta(R)-R$.*

Theorem 7. *Let $\{\mathfrak{U}_\alpha\}$ be a regular T-uniformity of a T_1-space R which agrees with the topology. A necessary and sufficient condition for R to be complete with respect to $\{\mathfrak{U}_\alpha\}$ is that $R = \Pi H_\alpha$ in $w(R)$, where H_α are defined by* (9).

Proof is obvious by Theorem 5.

Theorem 8. *In case $\{\mathfrak{U}_\alpha\}$ is a completely regular T-uniformity, we can replace $w(R)$ in Theorem 5 or 6 by $\beta(R)$.*

Proof. We have only to prove that $S(X_\lambda, \mathfrak{U}_\gamma) \subset U_\alpha$ implies $\bar{X}_\lambda \subset \tau(U_\alpha)$ in $\beta(R)$ (cf. the proof of Theorem 5), but this follows immediately from the fact that $\{U_\alpha, R-\bar{X}_\lambda\}$ is a normal covering of R.

Remark 1. In Theorems 5–8 it is sufficient to assume that R is a T-space.

Remark 2. As an application of Theorems 7 and 8 we can prove a theorem of E. Čech that a metrizable space R is complete with respect to some metric if and only if R is a G_δ-set in $\beta(R)$.

The "only if" part is obvious from Theorems 7 and 8. Let $\{\mathfrak{B}_m; m = 1, 2, \ldots\}$ be a completely regular T-uniformity of R agreeing with its topology and $R = \prod_{n=1}^{\infty} G_n$ with open sets G_n of $\beta(R)$. By the full normality of R we can easily construct a completely regular T-uniformity $\{\mathfrak{U}_n\}$ such that $\mathfrak{U}_n$ is a refinement of $\mathfrak{B}_n$ and $\sum_{U \in \mathfrak{U}_n} \tau(U) \subset G_n$. Then R is complete with respect to $\{\mathfrak{U}_n\}$ by Theorems 7 and 8.[5]

5) For a detailed proof cf. K. Morita, On the topological completeness, Shijo Sugaku Danwa-kai, 2nd ser., **13**, Jan. 1949. Cf. also J. Nagata, On topological completeness, Jour. Math. Soc. Japan, **2** (1950), p. 44.