36. *Remark on a Set of Postulates for Distributive Lattices.*

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1. Introduction.

G. Birkhoff gives the following set of postulates for distributive lattices:(*)

Any algebraic system which satisfies

(1) $a \frown a = a$ *for all* a ,

(2) $a \smile I = I \smile a = I$ *for some* I *and all* a ,

(3) $a \frown I = I \frown a = a$ *for some* I *and all* a ,

(4) $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$

and $(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$, *for all* a , b , c ,

is a distributive lattice with I .

G. Birkhoff proposes as the Problem 65 l.c. the following question: *Prove or disprove the independence of the seven identities assumed as postulates in Theorem 3.*

We shall remark first, that the system of axioms, as given above, is not sufficient to define the distributive lattices. If, indeed, we denote with I_2 one of the elements I in (2) and with I_3 one of I in (3), it may happen that $I_2 \neq I_3$, as the following example shows:

$\smile$	I_2	I_3
I_2	I_2	I_2
I_3	I_2	I_3

$\frown$	I_2	I_3
I_2	I_2	I_2
I_3	I_2	I_3

this system satisfies all the axioms (1)-(4), and is not a distributive lattice.

However, we may take sets of postulates, quite analogous to the one given above, to define a distributive lattices. We propose in the following lines four kinds of such postulate—sets, (I)-(IV). Any algebraic system, satisfying any one of these sets, turns out to be a distributive lattice with I. Each set consists of four, five or six postulates, which we shall prove as independent. Thus the Problem 65 of Birkhoff may be considered as solved.

2. Sets (I)–(IV).

Set (I)

(1) $a \frown a = a$ *for all* a,

(2)* $a \smile I = I \smile a = I$ *and* $a \frown I = I \frown a = a$, *for some* I *and all* a,

(4_1) $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$ *for all* a, b, c,

(4_2) $(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$ *for all* a, b, c.

Set (II)

(1) $a \frown a = a$ *for all* a,

(2)** $a \smile I = I \smile a = I$ *and* $a \frown I = a$, *for some* I *and all* a,

(3) $a \frown I = I \frown a = a$ *for some* I *and all* a,

(4_1) $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$ *for all* a, b, c,

(4_2) $(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$ *for all* a, b, c.

Set (III)

(1) $a \frown a = a$ *for all* a,

(2) $a \smile I = I \smile a = I$ *for some* I *and all* a,

(3)** $a \frown I = I \frown a = a$ *and* $a \smile I = I$, *for some* I *and all* a,

(4_1) $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$ *for all* a, b, c,

(4_2) $(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$ *for all* a, b, c.

Set (IV)

(1) $a \frown a = a$ *for all* a,

(2) $a \smile I = I \smile a = I$ *for some* I *and all* a,

(3) $a \frown I = I \frown a = a$ *for some* I *and all* a,

(4_1) $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$ *for all* a, b, c,

(4_2) $(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$ *for all* a, b, c,

(5)* $a \smile I = I$ *and* $a \frown I = a$ *for some* I *and all* a.

In each set the uniqueness of I is almost evident from (2)* of Set (I), from (2)** and (3) of Set (II), from (2) and (3)** of Set (III) and from (2), (3) and (5)* of Set (IV) respectively. Then the proof of Theorem 3 l.c.(*) will conclude easily that each of the sets (I)–(IV) defines a distributive lattice with I.

3. Consistency and Independence Proofs for Sets (I)–(IV).

The following system satisfies all the postulates in each of the sets (I)–(IV), and can be used as a formal consistency proof:

$\cup$	a	I
a	a	I
I	I	I

$\cap$	a	I
a	a	a
I	a	I

The independence of the postulates of sets (I)–(IV) is established by the following K_I–K_{IV}-system respectively: e.g. the system $K_I(1)$ satisfies all the postulates except the postulate (1) of Set (I).

$K_I(1)$, $K_{II}(1)$, $K_{III}(1)$, $K_{IV}(1)$

$\cup$	a	b	I
a	a	b	I
b	b	b	I
I	I	I	I

$\cap$	a	b	I
a	a	a	a
b	a	a	b
I	a	b	I

Here $a = b \cap b \neq b$.

$K_I(2)$*, $K_{II}(2)$**, $K_{III}(2)$, $K_{IV}(2)$

$\cup$	a	I
a	a	I
I	a	I

$\cap$	a	I
a	a	a
I	a	I

Here $a \cup I = I \neq I \cup a = a$.

$K_{II}(3)$, $K_{III}(3)$**, $K_{IV}(3)$

$\cup$	a	I
a	a	I
I	I	I

$\cap$	a	I
a	a	a
I	I	I

Here $a \cap I = a \neq I \cap a = I$.

$K_I(4_1)$, $K_{II}(4_1)$, $K_{III}(4_1)$, $K_{IV}(4_1)$

$\cup$	a	b	I
a	a	b	I
b	b	b	I
I	I	I	I

$\cap$	a	b	I
a	a	b	a
b	a	b	b
I	a	b	I

Here $a = a \cap (I \cup b) \neq (a \cap I) \cup (a \cap b) = b$.

$K_I(4_2)$, $K_{II}(4_2)$, $K_{III}(4_2)$, $K_{IV}(4_2)$

$\cup$	a	b	I
a	a	b	I
b	b	b	I
I	I	I	I

$\cap$	a	b	I
a	a	a	a
b	b	b	b
I	a	b	I

Here $a = (I \cup b) \cap a \neq (I \cap a) \cup (b \cap a) = b$.

$K_{IV}(5)$*

$\cup$	I_2	I_3
I_2	I_2	I_2
I_3	I_2	I_3

$\cap$	I_2	I_3
I_2	I_2	I_2
I_3	I_2	I_3

Here I_2, I_3 satisfy (2), (3) of Set (IV) respectively, and

$$I_3 \cup I_2 = I_2\,, \qquad I_2 = I_3 \cap I_2 \neq I_3$$

and

$$I_2 = I_2 \cup I_3 \neq I_3\,, \qquad I_2 \cap I_3 = I_2\,.$$

Reference.

(*) G. Birkhoff, LATTICE THEORY, American Mathematical Society, Colloquium Publication Vol. 25, Revised Edition, 1948, Theorem 3, Chapter IX, pp. 135-137.