51. On the Metrization and the Completion of a Space with Respect to a Uniformity.

By Jingoro SUZUKI.

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We first recall some definitions.[1] A collection $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ of open coverings of a topological space R is called a uniformity. If $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ satisfies the condition:

For any $\alpha, \beta \in \Omega$ there exists $\gamma \in \Omega$ such that $\mathfrak{U}_\gamma$ is a refinement of $\mathfrak{U}_\alpha$ and $\mathfrak{U}_\beta$, $\{\mathfrak{U}_\alpha\}$ is called a T-uniformity.

If $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ satisfies the condition:

For any $\alpha \in \Omega$ there exists $\lambda(\alpha) \in \Omega$ such that for each set $U_\lambda(\alpha) \in \mathfrak{U}_\lambda(\alpha)$ we can determine a set U_α of $\mathfrak{U}_\alpha$ and $\delta = \delta(\alpha, U_{\lambda(\alpha)}) \in \Omega$ so that $S(U_{\lambda(\alpha)}, \mathfrak{U}_\delta) \subset U_\alpha$, the uniformity $\{\mathfrak{U}_\alpha\}$ is called regular.

In § 1 we shall prove

Theorem 1. If a countable number of open coverings $\{\mathfrak{U}_n \mid n = 1, 2, \cdots\}$ of a T_1-space R forms a regular T-uniformity agreeing with the topology, then R is metrizable.

The simple extension R^* of a space R with respect to a uniformity $\{\mathfrak{U}_\alpha\}$ is not always complete. In § 2 we shall show that if we understand the notion of a Cauchy family in a more restricted sense, then the simple extension R^+ of R in this restricted sense is complete if $\{\mathfrak{U}_\alpha\}$ agrees with the topology of R.

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§ 1. Theorem 1 will be established by virtue of a theorem of A.H. Frink,[2] if the following three lemmas are proved.

Lemma 1. Under the assumption of the theorem there exists a uniformity $\{\mathfrak{V}_n \mid n = 1, 2, \cdots\}$ such that $\{\mathfrak{V}_n\}$ is equivalent to $\{\mathfrak{U}_n\}$ and $\mathfrak{V}_1 > \mathfrak{V}_2 > \cdots > \mathfrak{V}_n > \cdots$.

Proof. We put $\mathfrak{U}_1 = \mathfrak{V}_1$. Next we select $\mathfrak{U}_{\beta_2}$ such that $\mathfrak{U}_{\lambda(1)}, \mathfrak{U}_2 > \mathfrak{U}_{\beta_2}$ and put $\mathfrak{U}_{\beta_2} = \mathfrak{V}_2$. Now let us assume that $\mathfrak{V}_i$ are obtained for $i \leqq n$. We take $\mathfrak{U}_{\beta_{n+1}}$ such that $\mathfrak{U}_{\lambda(\beta_n)}, \mathfrak{U}_{n+1} > \mathfrak{U}_{\beta_{n+1}}$ and put $\mathfrak{U}_{\beta_{n+1}} = \mathfrak{V}_{n+1}$. Then $\{\mathfrak{V}_n \mid n = 1, 2, \cdots\}$ satisfies clearly the conditions of Lemma 1.

Lemma 2. For any point p of the space R and any index n, there exists an index m_0 such that

1) K. Morita: On the simple extension of a space with respect to a uniformity. I. Proc. Japan Acad. **27** No. 2, (1951).

2) A. H. Frink: Distance functions and the metrization problem. Bull. Amer. Math. Soc., vol. XLIII (1937), Theorem 4, p. 141.

$$S^2(p, \mathfrak{V}_{m0}) < S(p, \mathfrak{V}_n) .$$

We shall denote such m_0 by $\mu(p,n)$.

Proof. For the index n, we take the index $m_1 = \lambda(n)$ and for the index m_1, we take further the index $m_2 = \lambda(m_1)$. Then there is a V_{m2} such that $p \in V_{m2}$, $V_{m2} \in \mathfrak{V}_{m2}$. If for m_1, V_{m2} we take $l_1 = \delta(m_1, V_{m2})$, there exists some $V_{m1} \in \mathfrak{V}_{m1}$ satisfying the relation :

$$(1) \qquad S(V_{m2}, \mathfrak{V}_{l1}) < V_{m1} .$$

For the index n and V_{m1}, we take the index $l_2 = \delta(n, V_{m1})$, then for some $V_n \in \mathfrak{V}_n$ we have

$$(2) \qquad S(V_{m1}, \mathfrak{V}_{l2}) < V_n .$$

If we take finally $\mathfrak{V}_{m0}$ such that

$$(3) \qquad \mathfrak{V}_{l1}, \mathfrak{V}_{l2} > \mathfrak{V}_{m0} ,$$

then

$$S^2(p, \mathfrak{V}_{m0}) < S(p, \mathfrak{V}_n) .$$

In fact, from the relations (3), (1) and (2) it follows that

$$S^2(p, \mathfrak{V}_{m0}) = S(S(p, \mathfrak{V}_{m0}), \mathfrak{V}_{m0}) < S(S(p, \mathfrak{V}_{l1}), \mathfrak{V}_{l2}) < S(S(V_{m2}, \mathfrak{V}_{l1}), \mathfrak{V}_{l2})$$
$$< S(V_{m1}, \mathfrak{V}_{l2}) < V_n < S(p, \mathfrak{V}_n) .$$

This completes the proof of Lemma 2.

Lemma 3. For any point p of the space R and any index n, there exists some index $m = m(p, n)$ such that

$$(4) \qquad S(p, \mathfrak{V}_m) \frown S(q, \mathfrak{V}_m) \neq 0$$

implies

$$S(q, \mathfrak{V}_m) < S(p, \mathfrak{V}_n) .$$

Proof. For the index n, we put $m_0 = \mu(p, n)$ (cf. Lemma 2). For the index m_0, we put further $k = \mu(p, m_0)$. Then we have

$$(5) \qquad S^2(p, \mathfrak{V}_k) < S(p, \mathfrak{V}_{m0}) .$$

Moreover, if we take $\mathfrak{V}_m$ such that

$$(6) \qquad \mathfrak{V}_k, \mathfrak{V}_{m0} > \mathfrak{V}_m ,$$

then the index m satisfies the condition of this Lemma. Indeed by the relations (4), (6), (5) and Lemma 2 we have

$$S(q, \mathfrak{V}_m) < S^3(p, \mathfrak{V}_m) = S(S^2(p, \mathfrak{V}_m), \mathfrak{V}_m) < S(S^2(p, \mathfrak{V}_k), \mathfrak{V}_{m0})$$
$$< S(S(p, \mathfrak{V}_{m0}), \mathfrak{V}_{m0}) = S^2(p, \mathfrak{V}_{m0}) < S(p, \mathfrak{V}_n) .$$

This completes the proof of Lemma 3.

Thus we have proved that $\{S(p, \mathfrak{V}_n) | n = 1, 2, \cdots\}$ satisfies the conditions of A.H. Frink.[2] Hence the proof of Theorem 1 is completed.

Remark 1. L. W. Cohen said in his famous paper as follows: "The question arises as to whether a Hausdorff space, satisfying the first denumerability axiom and III_ω: To each $p \in R$ and n, there are positive integers $m(n)$ and $k(p, n)$ such that if $V_{k(p,n)}(q) \frown V_{m(n)}(p) \neq 0$ then $V_{k(p,n)}(q) \subset V_n(p)$, is metrizable." From the neighborhood system satisfying above conditions, we can construct a uniformity which satisfies the conditions of Theorem 1,[4] so that this question is affirmatively answered by our Theorem 1.[5]

§ 2. Let $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ be a uniformity of R. We shall say that a family $\{P_\lambda \mid \lambda \in \Lambda\}$ of subsets of R is a Cauchy family (with respect to the uniformity $\{\mathfrak{U}_\alpha\}$), if it has the finite intersection property and satisfies the condition:

1) For any $\alpha \in \Omega$ there exist a set $P_\lambda \in \{P_\lambda\}$ and $\beta \in \Omega$ and a set U_α of $\mathfrak{U}_\alpha$ such that $S^2(P_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$.

Theorem. 2. If $\{\mathfrak{U}_\alpha\}$ is a T-uniformity, then 1) is equivalent to the condition.

2) For any integer $n \geqq 2$ and α, there exist a set $P_\lambda \in \{P_\lambda\}$ and $\beta \in \Omega$ and a set U_α of $\mathfrak{U}_\alpha$ such that $S^n(P_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$.

Proof. It is evident that 2) implies 1). To show the converse it is sufficient to prove for $n = 3$. For the index α we take β and λ satisfying condition 1), and further for the index β we take λ_0 and β_0 such that it satisfies the condition 1). Let $\mathfrak{U}_{\beta_1}$ be a refinement of $\mathfrak{U}_\beta$ and $\mathfrak{U}_{\beta_0}$, then we have

$$S^3(P_{\lambda_0}, \mathfrak{U}_{\beta_1}) = S(S^2(P_{\lambda_0}, \mathfrak{U}_{\beta_1}), \mathfrak{U}_{\beta_1}) \subset S(U_{\beta,\beta}\, \mathfrak{U}_\beta) \subset S(S(P_\lambda, \mathfrak{U}_\beta), \mathfrak{U}_\beta)$$
$$\subset S^2(P_\lambda, \mathfrak{U}_\beta) \subset U_\alpha. \qquad \text{Q.E.D.}$$

It is to be noted that our definition of Cauchy family is more restrictive than that of K. Morita, that is $\{P_\lambda\}$ is a Cauchy family in the sense of K. Morita if it is a Cauchy family in our sense.

A Cauchy family $\{P_\lambda\}$ is said to be equivalent to another Cauchy family $\{Q_\mu\}$: written $\{P_\lambda\} \sim \{Q_\mu\}$, if for any $P_\lambda \in \{P_\lambda\}$ and any $\alpha \in \Omega$ there exist a set $Q_\mu \in \{Q_\mu\}$ and $\beta \in \Omega$ such that $S(Q_\mu, \mathfrak{U}_\beta) \subset S(P_\lambda, \mathfrak{U}_\alpha)$.

As is shown by K. Morita, the equivalence of Cauchy families is an equivalence relation.

For a subset A of R we denote by $\bar{A}^R$ the closure of A in R. We consider the equivalence classes of vanishing Cauchy families and denote by p^+ the class to which $\{P_\lambda\}$ belongs. Then we define

$$C = \{p^+ \mid \prod_\lambda \bar{P}_\lambda^R = 0 \text{ for some } \{P_\lambda\} \in p^+\},$$

3) L. W. Cohen: On imbedding a space in a complete space, Duke Math. Jour., vol. **5** (1939), p. 183.

4) Cf. K. Morita: Loc. cit.

5) Mr. M. Sugawara reported at the annual meeting of Math. Soc. of Japan in Oct. 1951 that he solved affirmatively this question of Cohen. His proof is not yet known to us.

and

$$R^+ = R + C\,.$$

Moreover, for any open set G of R we define the set G^+ as an open subset of $R+C$ as follows:

$G^+ = G+\{p^+ \mid p^+\in C$ and $\{P_\lambda\}\in p^+$ implies that $S(P_\lambda, \mathfrak{U}_\alpha) \subset G$ for some P_λ and $\mathfrak{U}_\alpha\}$, and define

$$\mathfrak{U}_\alpha^+ = \{U^+ \mid U \in \mathfrak{U}_\alpha\}\,.$$

We take G^+ as a basis of open sets of R^+. Then the following lemmas can be proved.[1]

Lemma 4. $\{\mathfrak{U}_\alpha^+\}$ is a uniformity of R^+.

Lemma 5. If $\{\mathfrak{U}_\alpha\}$ agrees with the topology of R, then $\{\mathfrak{U}_\alpha^+\}$ agrees with the topology of R^+.

Now we shall prove

Theorem 3. If $\{\mathfrak{U}_\alpha\}$ is a T-uniformity of R, agreeing with the topology of R, and $\{X_\lambda\}$ is a Cauchy family of R^+, then $\prod_{\lambda\in\Lambda} \bar{X}_\lambda^{R^+} \neq 0$.

Theorem 3 follows immediately from the following lemmas 6, 7 and 8.

Lemma 6. If we define $\{P_{\lambda\alpha} \mid P_{\lambda\alpha} = S(X_\lambda, \mathfrak{U}_\alpha^+)_\frown R,\ \lambda\in\Lambda,\ \alpha\in\Omega\}$, then $\{P_{\lambda\alpha} \mid \lambda\in\Lambda, \alpha\in\Omega\}$ is a Cauchy family of R. Accordingly it is also a Cauchy family in R^+.

Proof. Since it is clear that $\{P_{\lambda\alpha}\}$ has the finite intersection property (cf. Lemma 7 in Morita's paper), it is sufficient to prove that for any α there exist $P_{\lambda_0\alpha_0}$, $\beta\in\Omega$ and $U_\alpha\in\mathfrak{U}_\alpha$ such that $S^2(P_{\lambda_0\alpha_0}, \mathfrak{U}_\beta) \subset U_\alpha$.

For any index α, there exist $\lambda\in\Lambda$, $\beta\in\Omega$ and $U_\alpha^+ \in\mathfrak{U}_\alpha^+$ such that

$$(7) \qquad S^2(X_\lambda,\ \mathfrak{U}_\beta^+) \subset U_\alpha^+\,.$$

Next for the index β, there exist λ_0, β_0 and $U_\beta^+\in\mathfrak{U}_\beta^+$ such that

$$(8) \qquad S^2(X_{\lambda_0},\ \mathfrak{U}_{\beta_0}^+) \subset U_\beta^+\,.$$

Moreover, if $\mathfrak{U}_{\beta_1}$ is a refinement of $\mathfrak{U}_\beta$ and $\mathfrak{U}_{\beta_0}$, then we have $S^2(P_{\lambda_0\beta_1}, \mathfrak{U}_{\beta_1}) \subset U_\alpha$.

In fact, from (8) and (7) it follows that

$$S^2(P_{\lambda_0\beta_1}, \mathfrak{U}_{\beta_1}) \subset S^2(P_{\lambda_0\beta_1}, \mathfrak{U}_{\beta_1}^+) \subset S^2(S(X_{\lambda_0}, \mathfrak{U}_{\beta_1}^+), \mathfrak{U}_{\beta_1}^+) \subset S(S^2(X_{\lambda_0}, \mathfrak{U}_{\beta_0}^+), \mathfrak{U}_\beta^+)$$
$$\subset S(U_\beta^+, \mathfrak{U}_\beta^+) \subset S(S(X_\lambda, \mathfrak{U}_\beta^+), \mathfrak{U}_\beta^+) \subset S^2(X_\lambda, \mathfrak{U}_\beta^+) \subset U_\alpha^+\,.$$

Therefore $U_\alpha = U_\alpha^+{}_\frown R \supset S^2(P_{\lambda_0\beta_1}, \mathfrak{U}_{\beta_1})_\frown R = S^2(P_{\lambda_0\beta_1}, \mathfrak{U}_{\beta_1})$.

This completes the proof of Lemma 6.

Lemma 7. $\{X_\lambda\} \sim \{P_{\lambda\alpha}\}$, that is, for any λ and α there exist $P_{\lambda_0\alpha_0}$ and β_0 such that $S(P_{\lambda_0\alpha_0}, \mathfrak{U}_{\beta_0}^+) \subset S(X_\lambda, \mathfrak{U}_\alpha^+)$.

Proof. For any index α there exist λ_0, α_0 and $U_\alpha^+\in\mathfrak{U}_\alpha^+$ such that $S^2(X_{\lambda_0}, \mathfrak{U}_{\alpha_0}^+) \subset U_\alpha^+$. Then we have $S(P_{\lambda_0 \alpha_0}, \mathfrak{U}_{\alpha_0}^+) = S^2(X_{\lambda_0}, \mathfrak{U}_{\alpha_0}^+) \subset U_\alpha^+ \subset S(X_\lambda, \mathfrak{U}_\alpha^+)$.

Lemma 8. $\{X_\lambda\} \sim \{Y_\mu\}$ implies $\prod_\lambda \bar{X}_\lambda^{R+} = \prod_\mu \bar{Y}_\mu^{R+}$.

This is evident by Lemma 4 in Morita's paper, since $\{\mathfrak{U}_\alpha^+\}$ agrees with the topology of R^+.

Theorem. 4. If the uniformity $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ is a regular T-uniformity, the condition 1) of Theorem 2 is equivalent to the condition:

3) For any α there exist λ, β and $U_\alpha \in \mathfrak{U}_\alpha$ such that

$$S(P_\lambda, \mathfrak{U}_\beta) \subset U_\alpha.$$

Proof. It is evident that 1) implies 3). We shall prove the converse. For any index α, we take $\lambda(\alpha)$. By the condition 3) for $\lambda(\alpha)$ there exist λ, β_1 and $U_{\lambda(\alpha)} \in \mathfrak{U}_{\lambda(\alpha)}$ such that

$$S(P_\lambda, \mathfrak{U}_{\beta_1}) \subset U_{\lambda(\alpha)}. \tag{9}$$

For $\delta = \delta(\alpha, U_{\lambda(\alpha)})$ there exists $U_\alpha \in \mathfrak{U}_\alpha$ such that

$$S(U_{\lambda(\alpha)}\, \mathfrak{U}_\delta) \subset U_\alpha. \tag{10}$$

Moreover, we take $\mathfrak{U}_\beta$ such that $\mathfrak{U}_\delta, \mathfrak{U}_{\beta_1} > \mathfrak{U}_\beta$. Then from (10) and (9) it follows that

$$S^2(P_\lambda, \mathfrak{U}_\beta) \subset S(S(P_\lambda, \mathfrak{U}_{\beta_1}), \mathfrak{U}_\delta) \subset S(U_{\lambda(\alpha)}, \mathfrak{U}_\delta) \subset U_\alpha.$$

This completes the proof of Theorem 4.

Remark 2. Let R^* be the simple extension of R with respect to a uniformity $\{\mathfrak{U}_\alpha\}$ in the sense of K. Morita. Our extension R^+ is in general a subspace of R^*, but R^+ coincides with R^* for a regular T-uniformity, as is shown by Theorem 4.