50. On the Hauptsehne of the Region to which the Unit-Circle is Mapped by the Bounded Function.

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Let $F(z)(F(0)=0,\ F'(0)=1)$ be regular and schlicht in $|z|<1$ and B the mapped region of $|z|<1$ by $w=F(z)$. Then G. Szegö has proved that, for the length l of any Hauptsehne of the region B, the inequality,

$$l \geqq 1 \tag{1}$$

holds true.

In this note we will show that the relation (1) can be replaced by the more exact one, if we add the property that $F(z)$ is the bounded function, i. e.

$$|F(z)|<M \quad (M \geqq 1).$$

If r_1, r_2 are the boundary points on the Hauptsehne of the region B and lie on the opposite side with respect to the origin $w=0$, then

$$\arg r_2 = \arg r_1 + \pi.$$

Therefore,

$$l = |r_2 - r_1| = |r_1| + |r_2|,$$

where l denotes the distance of two points r_1, r_2.

The function

$$\phi(z) = \frac{r_1 M^2 F(z)}{[r_1 - F(z)][M^2 - \varepsilon r_1 F(z)]}, \quad |\varepsilon| = 1,$$

is normalized and regular, schlicht in $|z|<1$ and

$$\frac{M^2 r_1 r_2}{[r_1 - r_2][M^2 - \varepsilon r_1 r_2]}$$

is the boundary point of the mapped region of $|z|<1$ by $w=\phi(z)$. Hence, by the theorem due to Koebe, we have

$$\left| \frac{M^2 r_1 r_2}{(r_1 - r_2)(M^2 - \varepsilon r_1 r_2)} \right| \geqq \frac{1}{4}. \tag{2}$$

Being ε arbitrary, the inequality (2) is reduced to

$$\frac{M^2|r_1r_2|}{l(M^2+|r_1r_2|)} \geqq \frac{1}{4}$$

by putting $\varepsilon = -\frac{|r_1r_2|}{r_1r_2}$. And

$$\frac{|r_1r_2|}{M^2+|r_1r_2|} \leqq \frac{\left(\frac{|r_1|+|r_2|}{2}\right)^2}{M^2+\left(\frac{|r_1|+|r_2|}{2}\right)^2} = \frac{l^2/_4}{M^2+l^2/_4},$$

where the equality sign holds only when $|r_1| = |r_2|$. So we have

$$\frac{M^2 l}{4M^2+l^2} \geqq \frac{1}{4},$$

and then

$$l \geqq 2M(M-\sqrt{M^2-1}).$$

The equality sign holds true only when $|r_1| = |r_2|$ and

$$\frac{M^2|r_1r_2|}{l(M^2+|r_1r_2|)} = \frac{1}{4}.$$

Therefore, putting $\rho = M-\sqrt{M^2-1}(1+\rho^2 = 2M\rho)$, that

$$l = 2M(M-\sqrt{M^2-1}) = 2M\rho$$

is true only when

$$\frac{M^2 r_1 F(z)}{[r_1-F(z)]\left[M^2+\frac{|r_1r_2|}{r_2}F(z)\right]} = \frac{z}{(1+e^{i\alpha}z)^2} \tag{3}$$

and $|r_1| = |r_2| = M\rho$.
We know that the equality

$$\left|\frac{z}{(1+e^{i\alpha}z)^2}\right| = \frac{1}{4}$$

occurs only when $Z = e^{-i\alpha}$. Hence, by (3),

$$\frac{M^2 r_1 r_2}{(r_1-r_2)(M^2+|r_1r_2|)} = \frac{1}{4}e^{-i\alpha}.$$

From this equality, we have, by putting $r_2 = -r_1 = M\rho e^{i\theta}$,

$$e^{i\theta} = e^{-i\alpha}.$$

Thus it is concluded, from (3), that $F(z)$ should be the function defined by

$$\frac{M^2\rho e^{-i\alpha}F(z)}{[M\rho e^{-i\alpha}+F(z)][M^2+M\rho e^{i\alpha}F(z)]}=\frac{z}{(1+e^{i\alpha}Z)^2}, \tag{4}$$

where $l=2M\rho$.

By putting $e^{i\alpha}z=Z$ and $e^{i\alpha}F(z)=W$, (4) is reduced to

$$\frac{M^2\rho W}{(M\rho+W)(M+\rho W)}=\frac{Z}{(1+Z)^2},$$

and we have, by using the relation $1+\rho^2=2M\rho$,

$$ZW^2-M^2(1+Z^2)W+ZM^2=0,$$

or

$$F(z)=M\mu\frac{z^2+2\mu\frac{z}{M}+\mu^2-\sqrt{(z^2+\mu^2)^2-4\mu^2\frac{z^2}{M^2}}}{z^2+2\mu\frac{z}{M}+\mu^2+\sqrt{(z^2+\mu^2)^2-4\mu^2\frac{z^2}{M^2}}},\quad |\mu|=1.$$

Thus we can establish the following

Theorem.

Let $F(z)(F(0)=0,\ F'(0)=1)$ be regular and schlicht in $|z|<1$, and

$$|F(z)|<M,\quad (M\geqq 1),\quad \text{for } |z|<1,$$

then, the length l of any Hauptsehne of the mapped region to which the unit-circle $|z|<1$ is mapped by $w=F(z)$, satisfies the relation

$$l\geqq 2M(M-\sqrt{M^2-1}).$$

And the equality sign holds true for the function

$$F(z)=M_\mu\frac{z^2+2\mu\frac{z}{M}+\mu^2-\sqrt{(z^2+\mu^2)^2-4\mu^2\frac{z^2}{M^2}}}{z^2+2\mu\frac{z}{M}+\mu^2+\sqrt{(z^2+\mu^2)^2-4\mu^2\frac{z^2}{M^2}}},\quad |\mu|=1.$$

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