49. A Theorem of Liouville's Type for Meson Equation.

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In connection with the stochastic integrability of Fokker-Planck's equation[1], the author encountered with the following

Problem. Let R be a connected domain with smooth boundaries ∂R of an n-dimensional euclid space $R_n (n \geqq 2)$. Does there exist, for $m > O$, a bounded solution $h(x)$ other than 0 of the meson equation

$$(1) \qquad \Delta h(x) = m\,h(x) \text{ in } R$$

with the boundary condition

$$(2) \qquad \frac{\partial h}{\partial n} = O \text{ on } \partial R,\ n \text{ denoting outer normal?}$$

The purpose of the present note is to give an answer to this problem of Liouville's type. It reads as follows:

Let the boundaries ∂R lie entirely in the bounded part of R_n, and let $m(x)$ be a continuous function such that

$$(3) \qquad \inf_{x \in R} m(x) = m > 0.$$

Then the solution $h(x)$ of

$$(1) \qquad \Delta h(x) = m(x)\,h(x) \text{ in } R$$

satisfying the boundary condition (2) together with the order relation

$$(4) \qquad h(x) = O\left(\exp\left(\alpha|x|\right)\right), \text{ where } 2\alpha < \sqrt{2m}\,,$$

must vanish identically. Here $|x|$ denotes the distance of the point x from the origin of R.

Proof. 1st case (R is a bounded domain). By (2) and the Green's integral theorem, we have

$$m \int_R h(x)^2\,dx \leqq \int_R h(x)\,\Delta h(x)\,dx \leqq \int_{\partial R} h(x) \frac{\partial h}{\partial n}\,dS = O\,.$$

Here dx and dS respectively denote the volume element of R and the hypersurface element of ∂R. Thus $h(x)$ must $\equiv O$.

1) K. Yosida: Integration of Fokker-Planck's equation with a boundary condition, Journal of the Mathematical Society of Japan, Takagi's Congratulation Number. The result obtained below implies the existence of the Brownian motion in R with ∂R as reflecting barrier.

2nd case (R extends to infinity). Let r_0 be so large that the sphere K_{r_0} of radius r_0 with the origin as its centre contains ∂R entirely. Applying Green's integral theorem for the domain D_r bounded by the hypersurface ∂K_r of K_r and by ∂R, we have, by (2),

$$(5) \qquad m\int_{D_r} h(x)^2\,dx \leq \int_{D_r} h(x)\,\Delta h(x)\,dx \leq \int_{\partial K_r} h(x)\frac{\partial h}{\partial r}\,dS, \text{ for } r>r_0.$$

If we put

$$(6) \qquad F(r) = \int_{Dr} h(x)^2\,dx,$$

we have

$$F'(r) = \int_{\partial K_r} h(x)^2\,dS,\ F''(r) = \int_{\partial K_r} \frac{\partial h^2}{\partial r}\,dS + \int_{\partial K_r} h(x)^2 \frac{dS}{dr}.$$

Thus, by (5),

$$2\text{m}\ F(r) \leq F''(r) \text{ for } r > r_0.$$

Multiplying by $F'(r)$ and integrating from r_0 to r, we obtain

$$2\text{m}\ (F(r)^2 - F(r_0)^2) \leq F'(r)^2 - F'(r_0)^2.$$

If there exists $r_1 > r_0$ such that $F(r_1) > F(r_0)$, we have

$$\int_{r_1}^{r} \frac{dF}{\sqrt{2\text{m}(F^2 - F(r_0)^2) + F'(r_0)^2}} \geq \int_{r_1}^{r} dr.$$

Hence $\lim_{r\to\infty} F(r) = \infty$ and

(7) $F(r)$ is of order not smaller than $\exp(\sqrt{2\text{m}}\ r)$ as $r\to\infty$.

This contradicts to the assumption (4) and the definition (6) of $F(r)$. Thus $F(r) = F(r_0)$ for all $r > r_0$. Hence $h(x)$ must $= O$ for $|x| > r_0$. Therefore we obtain $h(x) = O$ in R by the same argument as in the 1st case.