47. *A Condition for an Abelian Group to be a Free Abelian Group with a Finite Basis.*

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1. Let G be a countable abelian (additive) group. An integer-valued function $f(\xi, \eta)$ on $G \times G$ is said to be bilinear if

$$f(\xi_1+\xi_2,\ \eta_1+\eta_2) = \sum_{i,j} f(\xi_i,\ \eta_j)$$

for any elements ξ_i, η_j, $(i, j = 1, 2)$.

Put

$$G_p = \{p\xi\ ;\ \xi \in G\},$$

where p is a fixed integer.

Prof. Igusa conjectured that *an abelian group G is a free abelian group with a finite basis if a bilinear integer-valued function $f(\xi, \eta)$ is defined on $G \times G$ which vanishes only at the identity element of $G \times G$, and if G/G_p is a finite group.*

The purpose of this note is to give an affirmative answer. We shall prove the

Theorem. *An abelian group G satisfying the above conditions is a free abelian group with a finite basis.*

2. From the above condition concerning the bilinear function $f(\xi, \eta)$ we can easily deduce that

(i) *there does not exist an element of finite order,*

(ii) *there does exist only a finite number of elements which are the divisors of a fixed element.*

An element is said to be prime if the element has not any divisor except itself. The set of prime elements $\{\xi_i\}$ is called a canonical system if for relative prime integers a_i the element

$$a_1\xi_1 + a_2\xi_2 + \cdots + a_s\xi_s$$

is also prime. A subgroup which is spanned by a canonical system is clearly a free abelian group. We shall now prove the following lemma.

Lemma. Let Ξ be a subgroup of rank $s-1$, spanned by a canonical system $\{\xi_1, \xi_2, \ldots, \xi_{s-1}\}$. If $G \neq \Xi$ there exists a canonical system $\{\varphi_1, \varphi_2, \ldots, \varphi_s\}$ such that the subgroup Φ spanned by the $\{\varphi_i\}$ contains Ξ and the rank of Φ is equal to s.

Proof of the Lemma. Let ξ_s be a prime element which is not contained in Ξ. If $\{\xi_1, \ldots, \xi_{s-1}, \xi_s\}$ is not a canonical system, then there exists a prime element η_1 such that for an integer $l_1 \neq 1$ and for relative prime integers a_i

$$a_1\xi_1 + \cdots + a_s\xi_s = l_1\eta_1 .$$

We can construct a unimodular matrix (a_{ij}) of order s whose first row coincides with $(a_1, \ldots, a_s)$ and whose components are all integers.

Put

$$a_{i1}\xi_1 + \cdots + a_{is}\xi_s = l_i\eta_i ,$$

where η_i's are all prime.

Then by simple calculations

$$\begin{aligned}(det(a_{ij}))^2 det(f(\xi_i, \xi_j)) &= det(f(l_i\eta_i, l_j\eta_j)) \\ &= (\Pi\, l_k)^2 det(f(\eta_i, \eta_j)).\end{aligned}$$

Or
$$det(f(\xi_i, \xi_j)) = (\Pi\, l_k)^2 det(f(\eta_i, \eta_j)) .$$

We have

$$det(f(\xi_i, \xi_j)) \neq 0 ,$$

because ξ_i's are linearly independent.

It follows that

$$|\, det(f(\xi_i, \xi_j))\,| > |\, det(f(\eta_i, \eta_j))\,| \geqq 1 .$$

If $\{\eta_i\}$ is not canonical we can construct a system $\{\zeta_i\}$ such that

$$|\, det(f(\eta_i, \eta_j))\,| > |\, det(f(\zeta_i, \zeta_j))\,| \geqq 1 .$$

Repeating this process we shall finally obtain a canonical system $\{\varphi_1, \ldots, \varphi_s\}$ such that the subgroup Φ spanned by $\{\varphi_i\}$ contains Ξ and has a rank s.

3. **Proof of the theorem.**[1]

Put

$$\Xi_p = \{p\xi\, ;\ \xi \in \Xi\} .$$

Then clearly

$$\Xi_p = G_p \frown \Xi .$$

By x we denote the rank of Ξ. Then

1) The technic of this step is due to Prof. Igusa. The author's proof is more complicated.

$$x^p = \text{the order of } \Xi/\Xi_p = \text{the order of } (\Xi+G_p)/G_p \leqq \text{the order of } G/G_p\,.$$

We see therefore the rank of such a Ξ is bounded, and by the above lemma G must coincide with one of such a Ξ.

Remark 1. It is true that without the condition that G/G_p is a finite group, G should be a free abelian group of infinite rank. The proof is almost analogous.

Remark 2[2]. An example of an abelian group which satisfies the conditions (i) and (ii), but is not a free abelian group is given by Prof. L. Pontrjagin: *Theory of topological commutative groups,* Appendix I. Annals of Math. vol. 35 (1934).

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2) Another kind of counter-examples were given by Prof. Akizuki.