63. *On the Possibility of the Weil's Integral Representation.*

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1. In the space of n complex variables $(z_1, \ldots, z_n)$, we take a closed domain P contained in a domain D. If there exist a finite number of functions $\varphi_1, \ldots, \varphi_m$ regular in D such that $|\varphi_j((z))| < 1$, $(j = 1, \ldots, m)$ for all points (z) in the interior of P, while at each boundary point of P at least one of them takes absolute unit value, then P is called a polyhedral domain in D.[1] We also assume that $m \geqq n$, and the varieties on its boundary

$$(1) \qquad \sigma_{j_1, \ldots, j_n} = \{(z) : |\varphi_{j_\mu}((z))| = 1 \,, \ (\mu = 1, \ldots, n)\}$$

have all at most dimension n. This condition shall be satisfied by suitable translations.

Next we assume[2] that there exist functions $p_{jk}((\zeta); (z))$, $(j = 1, \ldots, m; k = 1, \ldots, n)$ regular on $(\zeta) \in P$ and $(z) \in P$, satisfying

$$(2) \qquad \varphi_j((\zeta)) - \varphi_j((z)) = \sum_{k=1}^{n} (\zeta_k - z_k)\, p_{jk}((\zeta); (z)) \,.$$

We put

$$\Delta_{j_1 \cdots j_n}((\zeta); (z)) = \frac{\begin{vmatrix} p_{j_1 1}, \ldots\ldots, p_{j_n 1} \\ \cdots \qquad \cdots \\ p_{j_1 n}, \ldots\ldots, p_{j_n n} \end{vmatrix}}{\prod_{\mu=1}^{n} \left[\varphi_{j_\mu}((\zeta)) - \varphi_{j_\mu}((z))\right]}.$$

Under these conditions, a function $f(z_1, \ldots, z_n)$ regular on P[3] is represented by Weil's integral formula in P[4]

$$(3) \quad f((z)) = \frac{1}{(2\pi i)^n} \sum \int_{\sigma_{j_1 \cdots j_n}} f((\zeta)) \cdot \Delta_{j_1 \cdots j_n}((\zeta); (z)) d\zeta_1 \ldots d\zeta_n$$

1) Cf. for example: S. Hitotumatu, Cousin problems for ideals and the domain of regularity, will appear in Kōdai Math. Sem. Reports, vol. **3** (1951).

2) We will discuss this assumption later.

3) This means that f is regular in some neighborhood containing the closure of P.

4) A. Weil: Sur les séries de polynomes de deux variables complexes, C. R. Paris **194** (1932), 1304-5; L'intégrale de Cauchy et les fonctions de plusieurs variables, Math. Ann. **111** (1935), 178-182.

where the summation is taken over all n-combinations $(j_1, \ldots, j_n)$, where the indices are chosen from $(1, \ldots, m)$.

2. This integral formula is based upon the existence of the functions $p_{jk}((\zeta); (z))$ satisfying (2). Mr. A. Weil himself assumed it *a priori*. Later Mr. K. Oka[5] proved that there exist functions $R((\zeta); (z))$ and $p^*_{jk}((\zeta); (z))$, which are all regular on $(\zeta) \in P$ and $(z) \in P$, $R((\zeta); (\zeta)) \equiv 1$ and satisfying

$$(2') \quad R((\zeta); (z)) \cdot \left[\varphi_j((\zeta)) - \varphi_j((z))\right] = \sum_{k=1}^{n} (\zeta_k - z_k) p^*_{jk}((\zeta); (z)).$$

But this result is rather complicated because of the factor $R((\zeta); (z))$.

Recently, Mr. H. Hefer[6] showed that *there always exist functions $p_{jk}((\zeta); (z))$ satisfying* (2). His proof is elementary, but here, we remark that this result is easily proved by using the *theory of ideals* of analytic functions due to Messrs. K. Oka and H. Cartan.[7]

One of their important results is the following:[8]

Lemma.[9] *Suppose that an ideal $\mathfrak{J}$ with finite bases in a polyhedral domain P and a function $f(z_1, \ldots, z_n)$ regular on P satisfy the condition that for every point α of P, f belongs to the ideal $\mathfrak{J}_\alpha$ generated by $\mathfrak{J}$ at α. Then f belongs to $\mathfrak{J}$ itself.*

3. Now we will prove the above Hefer's result by using this lemma.

The closed domain $Q = P \times P$ in the space of $2n$ complex variables $\zeta_1, \ldots, \zeta_n; z_1, \ldots, z_n$ is also a polyhedral domain in $D \times D$. We consider there, the ideal $\mathfrak{J}$ with finite bases $\zeta_k - z_k$, $(k = 1, \ldots, n)$. Any point α of Q has the form $(a_1, \ldots, a_n; b_1, \ldots, b_n)$ where $(a), (b) \in P$. If $(a) \neq (b)$, the ideal $\mathfrak{J}_\alpha$ generated by $\mathfrak{J}$ at α is the unit-ideal, i.e., the family consisting of all functions regular at α. Therefore it is evident that $\mathfrak{J}_\alpha$ contains the functions $\varphi_j((\zeta)) - \varphi_j((z))$, $(j = 1, \ldots, m)$. If $(a) = (b)$, there exists a neighborhood

$$U = \{(\zeta; z); |\zeta_k - a_k| < \varepsilon, |z_k - a_k| < \varepsilon, (k = 1, \ldots, n)\}.$$

5) K. Oka: L'intégrale de Cauchy, Jap. J. of Math. **17** (1941), 523–531.

6) H. Hefer: Über eine Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung, Math. Ann. **122** (1950), 276–9.

7) K. Oka: Sur quelques notions arithmétiques, Bull. Soc. Math. France **78** (1950), 1–27; H. Cartan, Idéaux et modules de fonctions analytiques de variables complexes, Bull. Soc. Math. France **78** (1950), 29 64. See also Hitotumatu, loc. cit. 1).

8) For the terminologies used here, see Hitotumatu, loc. cit. 1).

9) "Théorème 4 bis" in Cartan, loc. cit. 7); see also Hitotumatu, loc. cit. 1) "Lemma 5a".

in which all the $\varphi_j\,(\zeta))$ and $\varphi_j((z))$ are regular. We now obtain the following expression in U:

$$\text{(4)} \qquad \varphi_j((\zeta))-\varphi_j((z)) = \sum_{k=1}^{n}(\zeta_k - z_k)\cdot\gamma_{jk}((\zeta)\,;\,(z))$$

where

$$\gamma_{jk}((\zeta)\,;\,(z)) = \frac{1}{\zeta_k - z_k}\Big[\varphi_j(z_1,\,\ldots,\,z_{k-1},\,\zeta_k,\,\zeta_{k+1},\,\ldots,\,\zeta_n)$$
$$-\varphi_j(z_1,\,\ldots,\,z_{k-1},\,z_k,\,\zeta_{k+1},\,\ldots,\,\zeta_n)\Big].$$

These functions $\gamma_{jk}((\zeta)\,;\,(z))$ are evidently regular in U except on the variety $\zeta_k = z_k$. But if ζ_k tends to z_k, $\gamma_{jk}((\zeta)\,;\,(z))$ converges to the function

$$\frac{\partial\varphi_j}{\partial z_k}(z_1,\,\ldots,\,z_k,\,\zeta_{k+1},\,\ldots,\,\zeta_n)$$

and so $\gamma_{jk}((\zeta)\,;\,(z))$ is bounded in U. Hence by the theorem on the removable singularities[10], $\gamma_{jk}((\zeta)\,;\,(z))$ is also regular on the variety $\zeta_k = z_k$. Therefore (4) means that the function $\varphi_j'(\zeta))-\varphi_j((z))$ belongs to the ideal $\mathfrak{J}_\alpha$ generated by $\mathfrak{J}$ at the point $\alpha = ((a)\,;\,(a))$. Therefore the hypothesis of the above Lemma is satisfied, and so $\varphi_j((\zeta))-\varphi_j((z))$ belongs to the ideal $\mathfrak{J}$ itself. This means the existence of $p_{jk}((\zeta)\,;\,(z))$ which are regular on Q and satisfy (2). Thus our assertion is proved.

10) Cf. for example: S. Bochner—W. T. Martin, Several Complex Variables, Princeton 1948, Chap. VIII, § 9.