60. *Theorems on the Cluster Sets of Pseudo-Analytic Functions.*

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(Comm. by K. KUNUGI, M.J.A., June 12, 1951.)

Let D be a domain on the z-plane and C be its boundary. Let E be a bounded closed set of capacity[1] zero, included in C and z_0 be a point in E. Let $w=f(z)$ be a single-valued function pseudo-analytic in D. The cluster set $S_{z_0}^{(D)}$ is the set of all values α such that $\alpha=\lim_{n\to\infty} f(z_n)$, where $z_n\ (n=1, 2, \dots)$ is a sequence of points tending to z_0 inside D. The cluster set $S_{z_0}^{*(C)}$ is the intersection of the closure of the union $US_{z'}^{(D)}$ for all z' belonging to the part of $C-E$, which lies in $|z-z_0|<r$.

Since E is of capacity zero, by Evan's theorem[2], we can distribute a positive measure $d\mu(a)$ on E such that its potential

$$u(z)=\int_E \log\frac{1}{|z-a|}d\mu(a), \qquad \int_E d\mu(a)=1$$

is harmonic outside E, excluding $z=\infty$, and has boundary value $+\infty$ at any point of E. Let $v(z)$ be its conjugate harmonic function and put

$$\zeta=\zeta(z)=e^{u(z)+iv(z)}=r(z)e^{iv(z)}=re^{i\theta}.$$

The niveau curve $C_r: r(z)=\text{const.}=r\ (0<r<+\infty)$ consists of a finite number of Jordan curves surrounding E. Let J_r be its component which surrounds z_0. Let V_r be the closure of the set of all values taken by $f(z)$ in the part of D, which lies in the interior of J_r. Then $S_{z_0}^{(D)}$ is identical with the intersection of all V_r. Let M_r be the closure of the union $US_{z'}^{(D)}$ for all z' belonging to the part of $C-E$, which lies in the interior of J_r. Then $S_{z_0}^{*(C)}$ is identical with the intersection of all M_r. *Let (P) denote the class of functions $w=f(z)$ which are single-valued and pseudo-analytic in D and for which the integral*

$$\int^{\infty}\frac{dr}{rD(r)} \tag{1}$$

diverges, where $D(r)$ is the smallest upper bound of the 'Dilatationsquotient'[3] $D_{z|w}$ of $w=f(z)$ on the part of C_r which lies in D.

1) 'Capacity' means logarithmic capacity in this paper.
2) G. C. Evans: Monatshefte f. Math. u. Phys. 43 (1936).
3) O. Teichmüller: Deutsche Math. 3 (1938).

Let G be a domain on the w-plane bounded by a Jordan curve Γ and a bounded closed set F. We introduce a Riemannian metric[4)]

$$ds = \lambda(w)\,|dw| \tag{2}$$

on G, where $\lambda(w)$ is a non-negative, continuous function in G such that the metric gives G a finite area.

Lemma 1. *Let $w=f(z)$ be a function of (P) and Δ be a subdomain of D such that its boundary does not contain any point of $C-E$ and any value taken by $f(z)$ in Δ lies in G. Let $A(r)$ be the area of the Riemannian image of the part of Δ, which lies between C_r and C_{r_0} and $L(r)$ be the length of the image of the part of C_r, which lies in Δ. Then we have*

$$\lim_{r\to+\infty}\frac{L(r)}{A(r)}=0\,. \tag{3}$$

Proof. Let C_r' be the part of C_r, which lies in Δ and θ_r be its image on the ζ-plane by $\zeta=\zeta(z)$. Let $z=z(\zeta)$ be the inverse function of $\zeta=\zeta(z)$ and put $w(\zeta)=f(z(\zeta))$. If we denote the differential coefficient of $w(\zeta)$ along θ_r by w', then we have

$$L(r)=\int_{\theta_r}\lambda(w(\zeta))\,|w'|\,rd\theta\,.$$

Hence, by the inequality of Schwarz, we have

$$(L(r))^2\leqq\int_{\theta_r}rd\theta\int_{\theta_r}\lambda^2|w'|^2rd\theta\leqq 2\pi r\int_{\theta_r}\lambda^2|w'|^2rd\theta\,.$$

Since $D_{z|w}=D_{\zeta|w}$, we have

$$\frac{1}{2\pi}\int_{r_1}^{r}\frac{(L(r))^2}{rD(r)}dr\leqq\int_{r_1}^{r}\int_{\theta_r}\frac{\lambda^2|w'|^2}{D(r)}rd\,rd\theta\leqq A(r)-A(r_1)\,, \tag{4}$$

where $r>r_1$. Letting $r_1\to r$, we have

$$\frac{dr}{2\pi rD(r)}\leqq\frac{dA(r)}{(L(r))^2}\,.$$

Let Δ_r be the set of all values r such that $L(r)>\sqrt{A(r)}\log A(r)$, then we have

$$\frac{1}{2\pi}\int_{\Delta_r}\frac{dr}{rD(r)}\leqq\int_{\Delta_r}\frac{dA(r)}{(L(r))^2}\leqq\int_{A(r_0)}^{\infty}\frac{dt}{t(\log t)^2}<+\infty$$

4) L. Ahlfors: Acta Soc. Sci. Fenn. N. s. 2 (1937).

Since the integral (1) diverges, we have (3) in the case when $A(r)$ is not bounded. If $A(r)$ is bounded, then we have $\varliminf_{r\to+\infty} L(r) = 0$ by (4), so that we have (3).

Lemma 2. *If the set F is of capacity positive, then there exists a metric* (2) *which gives F a positive length. Suppose further that F is not covered by the closure of a finite covering surface W of G. If we denote the area and the length of the relative boundary of W by A and L respectively, then we have $A \leqq hL$, where h is a positive constant.*

Proof. Since F is a set of capacity positive, we can distribute a positive measure $d\mu(a)$ on F such that its potential

$$\xi(w) = \int_F \log\frac{1}{|w-a|}d\mu(a)\,, \qquad \int_F d\mu(a) = 1$$

is harmonic in the complementary domain $G(F)$ of F, which contains G, excluding $w=\infty$, and has boundary values not greater than the Robin's constant γ of $G(F)$[5]. Let $\eta(w)$ be its conjugate harmonic fnnction and put $\omega = \omega(w) = \exp\{\xi(w) + i\eta(w)\}$. The functions $|\omega(w)|$ and $|\omega'(w)|$ are single-valued. Let β be a Jordan curve or a finite number of Jordan curves surrounding F, then we have

$$\int_\beta d\eta(w) = 2\pi\int_F d\mu(a) = 2\pi\,.$$

Hence we can put $\lambda(w) = |\omega'(w)|/(1+|\omega(w)|^2)$ in (2). The area of G is not greater than π. Since $\xi(w)\leqq\gamma$ in G, the length of F is positive. Hence, by Ahlfors' theory of covering surfaces[6], we have $A\leqq hL$.

Lemma 3. *If a function $w=f(z)$ of* (P) *is bounded in D and*

$$\varlimsup_{z\to z'}|f(z)|\leqq M \tag{5}$$

for every point z' of $C-E$, then $|f(z)|\leqq M$ in *D.*

Proof. We suppose, contrary to the assertion, that there exists a point z_1 in D such that $|f(z_1)|>M$. Since $f(z)$ is bounded, there exists a constant K such that $|f(z)|<K$ in D. We have $K>M$. Let M_1 be a constant such that $|f(z_1)|>M_1>M$. We choose the domain G such that Γ is the circle $|w|=M_1$ and F is a bounded closed set of capacity positive lying outside the circle $|w|=K+1$. Then there exists a metric of Lemma 2. Let Δ be

5) R. Nevanlinna: Eindeutige analytische Funktionen (1936).

6) L. Ahlfors: Acta Math. 65 (1935).

the set of all points z in D such that $w=f(z)$ lies in G. Since $f(z_1)$ lies in G, $\varDelta$ is not empty. The boundary of $\varDelta$ does not contains any point of $C-E$ by (5). Let r_0 be a number such that z_1 lies in the interior of the niveau curve C_{r_0}. Let $A(r)$ be the area of the Riemannian image W_r of the part of $\varDelta$, which lies between C_r and C_{r_0} and $L(r)$ be the length of the image of the part of C_r, which lies in $\varDelta$, respectively by $w=f(z)$. Since the closure of W_r does not cover F, by Lemma 2, we have

$$A(r) \leqq h(L(r)+L(r_0)),$$

where h is a positive constant. Hence, by Lemma 1, $A(r)$ is bounded.

Let M_2 be a constant such that $|f(z_1)|>M_2>M_1$. We denote the circle $|w|=M_2$ by Γ', the domain bounded by Γ' and F by G' and the set of all points z in D such that $w=f(z)$ lies in G' by $\varDelta'$. If the closure of $\varDelta'$ is contained in D, then the Riemannian image of $\varDelta'$ by $w=f(z)$ is a finite covering surface of G', which has not relative boundary. Since the closure of this covering surface does not cover F, we arrive at a contradiction by Lemma 2, so that $\varDelta'$ contains at least a point of E on its boundary. Hence C_r meets the boundaries of $\varDelta$ and $\varDelta'$ for a sufficiently large r, so that we have $\varlimsup_{r\to+\infty} L(r)>0$. Hence, by Lemma 1, $A(r)$ is not bounded, which is a contradiction. Therefore $|f(z)|\leqq M$ in D.

Theorem 1. *If $w=f(z)$ is a function which belongs to the class (P), then $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is an open set. Suppose further that Ω is not empty, then $w=f(z)$ takes every values in Ω, except those belonging to a set of capacity zero, infinitely often in any neighbourhood of z_0.*

Proof. We choose the domain G bounded by a Jordan curve Γ and a closed set F such that its closure and $S_{z_0}^{*(C)}$ have no point in common and $S_{z_0}^{(D)}$ and G have at least a point in common. Since M_r is the closure of the union $US_{z'}^{(D)}$ for all z' belonging to the part of $C-E$, which lies in the interior of J_r, there is a number r_0 such that M_r and the closure of G have not any point in common for every $r\geqq r_0$. Let $D(G)$ be the set of all points z in D such that $w=f(z)$ lies in G. Then the boundary of $D(G)$ does not contain any point of $C-E$, which lies in the interior of J_{r_0}. Let w_0 be a point of $S_{z_0}^{(D)}$ contained in G. Then there exists a sequence of points z_n $(n=1, 2, \ldots)$ tending to z_0 inside D such that $w_0=\lim_{n\to\infty} f(z_n)$. We denote the component of $D(G)$, which contains z_n by $\varDelta_n$.

7) L. Ahlfors: Loc. cit. 6).

If there exists a component $\varDelta_k$ which contains infinitely many points z_n, then z_0 is a boundary point of $\varDelta_k$. In this case, we denote the part of $\varDelta_k$, which lies in the interior of J_{r_0} by $\varDelta$. If such a component does not exist, then the sequence $\{\varDelta_n\}$ contains infinitely many distinct components. Since the curve J_r does not meet infinitely many distinct components $\varDelta_n$ for every $r \geqq r_0$, $\varDelta_n$ is contained in the interior of J_r for a sufficiently large n, that is, the sequence $\{\varDelta_n\}$ tends to z_0. In this case we denote the union of all $\varDelta_n$ which lie in the interior of J_{r_0} by $\varDelta$. Let $\varDelta(r)$ be the part of $\varDelta$, which lies outside of C_r and W_r be its Riemannian image by $w=f(z)$. Let $A(r)$ be the area of W_r and $L(r)$ be the length of the image of the part of C_r, which lies in $\varDelta$. Then we have the same relation as (3) of Lemma 1.

Let G' be a subdomain of G, which contains w_0 and whose closure lies in G and $\varDelta'$ be the set of all points z in $\varDelta$ such that $w=f(z)$ lies in G'. If $\varDelta'$ contains a sequence of components tending to z_0, then the closure of the set of all values taken by $f(z)$ in a component of $\varDelta'$ is identical with the closure of G' by Lemma 3, so that $A(r)$ is not bounded. If $\varDelta'$ contains a component which has z_0 on its boundary, then C_r meets the boundaries of $\varDelta$ and $\varDelta'$ for a sufficiently large r, so that $\overline{\lim_{r\to+\infty}} L(r) > 0$. Hence $A(r)$ is not bounded. Therefore we have in all cases $\lim_{r\to+\infty} A(r) = +\infty$.

If we suppose, contrary to the assertion, that Ω is not an open set. Then we can choose the domain G such that F is a bounded closed set of capacity positive lying outside $S_{z_0}^{(D)}$. Since V_r is the closure of the set of all values taken by $f(z)$ in the part of D, which lies in the interior of J_r, there is a number r_1 such that V_r and F have not any point in common for every $r \geqq r_1$. We can choose r_0 such that $r_0 > r_1$. Then, by Lemma 2, there is a metric and a positive constant h such that

$$A(r) \leqq h(L(r)+L(r_0)) .$$

Hence we have

$$\frac{1}{h} \leqq \underline{\lim}_{r\to+\infty} \frac{L(r)+L(r_0)}{A r)} = 0 ,$$

which is a contradiction, so that Ω is an open set.

Let Ω_n be a component of Ω and F_n be the set of all values in Ω_n, which is ommitted by $f(z)$ in a neighbourhood of z_0. We choose the domain G such that its closure is contained in Ω_n and F is identical with F_n. Let r_1 be a number so large that J_r lies in this neighbourhood of z_0 for every $r \geqq r_1$. If we suppose that F_n is a set of capacity positive, then, by the same reason as above, we arrive at a contradiction. Hence F_n is a set of capacity zero, so that, by the well known method, we can prove that the set of exceptional values is of capacity zero.

Theorem 2. *If the set E is contained in a finite number of connected components of the boundary C and Ω is not empty, then $w=f(z)$ takes every values, with two possible exceptions, belonging to any connected component Ω_n of Ω infinitely often in any neighbourhood of z_0.*

Proof. We suppose, contrary to the assertion, that there are three exceptional values in Ω_n and denote the set of these values by F. Then there is a number r_1 such that $f(z)$ does not take any value of F in the part of D, which lies in the interior of J_{r_1}. We choose the domain G bounded by F and a Jordan curve Γ such that its closure is contained in Ω_n. Then there is a number r_2 such that M_r and the closure of G have not any point in common for every $r \geqq r_2$. We put $r_0 = \text{Max}(r_1, r_2)$ and use the proof of Theorem 1.

Let I be the area of G and put $A(r) = IS(r)$. When $\Delta(r)$ is a single domain, we denote its characteristic number by η and put $\eta^+ = \text{Max}(O, \eta)$. When $\Delta(r)$ consists of a finite number of connected components, we denote the sum of such numbers for every components by the same notation η^+. Since F consists of three points, we have by the fundamental theorem of Ahlfors[7]

$$\eta^+ \geqq 2S(r) - h(L(r) + L(r_0)), \tag{6}$$

where h is a positive constant.

Let $m(r)$ be the number of Jordan curves contained in the boundary of $\Delta(r)$, whose images by $w=f(z)$ lie on Γ. Then, by a method of Kunugi[8], we have

$$m(r) \leqq S(r) + h'(L(r) + L(r_0)), \tag{7}$$

where h' is a positive constant. Let $n(r)$ be the number of connected components of the union of C and the closures of the domains bounded by C_r, which contain a point of E. Then $n(r)$ is bounded and $\eta^+ \leqq m(r) + n(r)$, so that we have from (6) and (7)

$$1 \leqq (h+h')\frac{L(r)+L(r_0)}{S(r)} + \frac{n(r)}{S(r)}.$$

Since $A(r) = IS(r)$ is not bounded, we arrive at a contradiction by (3).

8) K. Kunugi: Proc. 16 (1940), Jap. Jour. of Math. 18 (1942).

Remark 1. Lemma 3 is an extension of a theorem which we have proved recently[9]. Theorem 1 is an extension of a theorem of Tsuji[10]. Theorem 2 contains the case when E consists of a single point and the case when D is simply connected, so that it is an extension of a theorem of Kunugi[11] and that of Noshiro[12].

Remark 2. Let $D(r)$ be a continuous function such that $D(r) \geqq 1$ for every $r \geqq r_0$. Then the function $w = f(z)$:

$$f(z) = e^{\zeta}, \ \zeta = \exp\left\{\int_{r_0}^{r} \frac{dr}{rD(r)} + i\theta\right\}, \quad z = \frac{1}{r} e^{-i\theta},$$

is single-valued and pseudo-analytic in the domain $0 < |z| < 1/r_0$. Its 'Dilatationsquotient' is equal to $D(r)$ at every points on the circle $|z| = 1/r$. If the integral (1) converges, then the function $w = f(z)$ is bounded. Hence (P) *is the maximal class* for which we can extend the theory of cluster sets.

9) T. Yosida: Proc. 26 (1950).
10) M. Tsuji: Proc. 19 (1943).
11) K. Kunugi: Loc. cit.
12) K. Noshiro: Jour. Math. Soc. Jap. 1 (1950), Nagoya Math. 1 (1950).