73. *On B*-Algebras.*

By Masanori FUKAMIYA.

(Comm. by K. KUNUGI, M.J.A., July 12, 1951.)

§ 1. *Introduction.* It can be shown that, for any B*-algebra R with the unit element, the following three conditions are equivalent:

(A). For every $x \varepsilon R$, $\neq 0$, there is a linear positive functional $f(x)$ such that $f(x^*x) > 0$.

(B). The set D_0 of all elements whose spectrum is positive is identical with the set $\mathfrak{D}_0 = \{u / f(u) > 0, \text{ for all } f \varepsilon P_0\}$, where P_0 denotes the set of all linear positive functional $f(x)$ on R, with $f(e) = 1$.

(C). For every $x \varepsilon R$, $x^*x + e$ has the inverse element.

A B*-algebra satisfying one of these conditions is called a C*-algebra. We assume here, that P_0 under consideration is not empty.

This paper continues the study of B*- and C*-algebras. We assume that P_0 is not empty. First, we shall prove that every B*-algebra with the unit element for which $K = 0$ are valid is a C*-algebra, that the quotient algebra of any B*-algebra with the unit modulo any maximal two-sided ideal is a (simple) C*-algebra. Gelfand and Neumark conjectured that every B*-algebra would be the C*-algebra, but this is reduced to the following questions: (1) Is there a B*-algebra for which P_0 is empty? (2) Is there a B*-algebra for which $K \neq (0)$ (or, $N(x) \neq \| x \|$)? We cannot answer to these questions here. [2, 6]

Next, we shall prove certain theorems on the ideals of B*- and C*-algebras, simply and without use of the representation by operators on the Hilbert space; some of these are originally due to I.E. Segal [1]. The method of proof of Lemmas 1 and 2 has already been shown by I. Kaplansky, but we repeat it here for the sake of completeness [6].

§ 2. *The Theorems.* A C*-algebra R with the unit element e is a normed ring, over the field of complex numbers, which satisfies the following conditions:

(a) An involutorial anti-automorphism $x \rightarrow x^*$ is defined on R, such that (i) $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$, (ii) $(xy)^* = y^* x^*$, (iii) $x^{**} = x$;

(b) $\| x^* x \| = \| x \|^2$, for every $x \varepsilon R$;

(c) $x^*x + e$ possesses the (two-sided) inverse for every $x \varepsilon R$;

(d) R has the unit element e and is complete in the norm.

A B*-algebra R with the unit element is a normed ring, over the field of complex numbers, which satisfies the conditions (a), (b) and (d).

A positive linear functional $f(x)$ on a B*-algebra is a complex-valued linear functional on the complex Banach space R such that $f(x^*x)\geqq 0$, for every $x\varepsilon R$. We denote by P_0 the set of all positive linear functionals on R with $f(e)=1$.

The spectrum of an element x of a B*-algebra R is the set of all complex numbers λ such that $x-\lambda e$ has no (two-sided) inverse.

Every element u with $u=u^*$ is called hermitian. The spectrum of an hermitian element of a B*-algebra consists of real numbers (Appendix, (2), below). If the spectrum of an hermitian element consists of non-negative numbers, the element is said positive. In every C*-algebra, the element of the form x^*x is positive, but it is not known for B*-algebras in general.

Lemma 1. For any closed left ideal I in a B*-algebra, $x^*x\varepsilon I$ implies $x\varepsilon I$ (or, $xx^*\varepsilon I$ implies $x^*\varepsilon I$).

If I is a closed right ideal, then $x^*x\varepsilon I$ implies $x^*\varepsilon I$ (or, $xx^*\varepsilon I$ implies $x\varepsilon I$).

Proof. Let I be any closed left ideal and let $x^*x\varepsilon I$. As x^*x is hermitian, $\varepsilon e+(x^*x)^2$ has the inverse for every $\varepsilon>0$. Take the hermitian element $w=(x^*x)^2\{\varepsilon e+(x^*x)^2\}^{-1}$. As x^*x commutes with $\{\varepsilon e+(x^*x)^2\}^{-1}$, $w\varepsilon I$. We show that $x=\lim\limits_{\varepsilon\to 0} xw$. For this, by (b), it would be sufficient to see that $\|x(w-e)\|^2=\|\{x(w-e)\}^*\{x(w-e)\}\|$ $=\|(w-e)\,x^*x\,(w-e)\|=\|(w-e)^2x^*x\|\to 0$, when $\varepsilon\to 0$. Let $R(u)$, $u=x^*x$, be the totality of elements which and whose adjoints commute with all those elements of R which commute with x^*x. $R(u)$ is a closed commutative B*-subalgebra of R, and contains x^*x, e and $\{\varepsilon e+(x^*x)^2\}^{-1}$. Then, a theorem on commutative B*-algebras can be applied, and we have $\|(w-e)^2x^*x\|=\varepsilon^2\|x^*x\{\varepsilon e+(x^*x)^2\}^{-2}\|=\varepsilon^2\cdot\sup\limits_{M\varepsilon\mathfrak{M}}|u(M)|\cdot[\varepsilon+\{u(M)\}^2]^{-2}$, $(u=x^*x)$, where M and $\mathfrak{M}$ denote the maximal (self-adjoint) ideal and the totality of maximal ideals of $R(u)$, respectively. As $\sup\limits_{M\varepsilon\mathfrak{M}}|u(M)|\cdot[\varepsilon+\{u(M)\}^2]^{-2}$ $\leqq\max\limits_{0<\beta<\infty}\beta/(\varepsilon+\beta)^{-2}=\dfrac{9}{16\sqrt{3}}\sqrt{\varepsilon}$, we have $\lim\limits_{\varepsilon\to 0}\|(w-e)^2x^*x\|=0$; so we have $\lim\limits_{\varepsilon\to 0} xw=x$ and $\lim\limits_{\varepsilon\to 0} wx^*=x^*$. Since $xw\varepsilon I$, as $w\varepsilon I$ and I is closed, we have $x\varepsilon I$.

If I is a right closed ideal and $x^*x\varepsilon I$, then wx^* and $x^*\varepsilon I$.

In the same manner for xx^*, we have Lemma 1.

Lemma 2. (Segal and Kaplansky) A closed two-sided ideal I in a B*-algebra is self-adjoint, that is: $x\varepsilon I$ implies $x^*\varepsilon I$.

Proof. Let x be any element of a two-sided closed ideal I. As $xx^*\varepsilon I$, Lemma 1 shows that $x^*\varepsilon I$.

Lemma 3. (Gelfand and Neumark) If $u\varepsilon R$ is any positive and regular element of a B*-algebra R, and v is any hermitian element, then $u+iv$ is regular.

Proof. The equation $(x+iy)(u+iv)=(u+iv)(x+iy)=e$, or $xu-yv=e$, $xv+yu=0$ has the hermitian solutions x, y. To see this, put $p=e+(vu^{-1})^2$, then $p=u_1(e+w^2)u_1^{-1}$, where $u=u_1^2$ and $w=u_1^{-1}pu_1^{-1}$. Now p is regular, and $x=u^{-1}p^{-1}$, $y=-u^{-1}p^{-1}(vu^{-1})$ are the solutions.

Lemma 4. (Gelfand and Neumark) If R is a C*-algebra with the unit, and if u, v are any positive elements, then $u+v$ is positive.

Proof. It must be proved in general that $u+v+e$ is regular. As $u'=u+e$ is positive and regular, it suffices to prove the following Lemma 4'.

Lemma 4'. (Gelfand and Neumark) If R is a C*-algebra with the unit, and if u is any positive and regular element and v any positive element, then the spectrum of uv (and vu) is real and non-negative.

Proof. That the spectrum of uv is real is proved without (c). In fact, let $u=u_1^2$, $v=v_1^2$, where u_1 is positive and regular and v is positive, then $uv=u_1(p^*p)u_1^{-1}$, $p=v_1u_1$; as p^*p has the real spectrum and uv and p^*p have the same spectrum, the spectrum of uv is real.

Next, by (c), $w=uv+e=u_1(p^*p+e)u_1^{-1}$ shows that $uv+e$ is regular, which completes the proof.

Let E_R be the real Banach space of all hermitian elements of a C*-algebra R, with the original norm in R, and let D and D_0 denote the sets (in E_R) of all positive and positive, regular elements of R, respectively. Then, by Lemma 4, D is a convex, conical body in E_R, and D_0 is the interior of D.

Let I be any closed left ideal of a C*-algebra R, and let H_I be the linear closed subspace of E_R spanned by hermitian components of all elements of I. On account of the openness of D_0 and of Lemma 3, we can easily find that H_I does not contain any point of D_0.

Lemma 5. For every element $x\neq 0$ of a C*-algebra R, there is a positive linear functional $f(x)$ such that $f(x^*x)=\|x\|^2$, $f(e)=1$.

Proof. In E_R, the elements of the form $\alpha e+\beta x^*x$, α and β being any real numbers, forms a linear, closed subspace, E_1. Define a functional $f_1(u)$ on E_1 such that $f_1(e)=1$, $f_1(x^*x)=\|x^*x\|$ and $f_1(u)=\alpha+\beta\|x^*x\|$, for $u=\alpha e+\beta x^*x$. It is easily seen that $f_1(u)$ is a positive linear functional on E_1. By the well-known method of extension of positive linear functionals and the preceding consideration, we are able to obtain a linear functional $f(x)$ on E_R such that $f(D)\geqq 0$ and $f(u)=f_1(u)$ for $u\varepsilon E_1$. Defining $f(y)=f(u)+if(v)$, for $y=u+iv\varepsilon R$, we have, since $y^*y\varepsilon D$ (by (c)), for every $y\varepsilon R$, a positive linear functional on R, which has the required property.

For any two-sided closed ideal I in a B*-algebra, on account of Lemma 2, an element x belongs to I if and only if each of its hermitian components belongs to I; and, on account of Lemma 1, an hermitian element u belongs to I if and only if u^2 belongs to I. As every positive hermitian element u has the form $u=v^2$ in a B*-algebra, where v is positive and hermitian, thus every hermitian element is the difference of two positive hermitian elements $\left(u_+ = \frac{v+u}{2},\ u_- = \frac{v-u}{2}\right)$; therefore, any two-sided closed ideal of a B*-algebra is generated by all its positive hermitian elements $u=v^2$, where $v \varepsilon I$.

Now, we can prove the following Lemma.

Lemma 6. For any two-sided closed ideal I of a B*-algebra R, it holds:

$$\inf_{z\varepsilon I} \| (x+z)^*(x+z) \| = \inf_{z\varepsilon I} \| x^*x+z \| .$$

Proof. For $x\varepsilon I$, this identity is trivially valid.

As $\inf_{z\varepsilon I} \| (x+z)^*(x+z) \| \geqq \inf_{z\varepsilon I} \| x^*x+z \|$ is clear, we will prove the converse inequality. Since $\| (x+z)^*(x+z) \| \leqq \| x^*x+z^*z \| + \| z^*x+x^*z \|$ and $z^*x+x^*z\varepsilon I$ holds for every two-sided ideal I, we have $\inf_{z\varepsilon I} \| (x+z)^*(x+z) \| \leqq \inf_{z\varepsilon I} \| x^*x+z^*z \| + \inf_{z\varepsilon I} \| z^*x+x^*z \| = \inf_{z\varepsilon I} \| x^*x + z^*z \|$. The preceding remark shows that $\inf_{z\varepsilon I} \| x^*x+z^*z \| = \inf_{\substack{z\varepsilon I \\ z \text{ hermitian} \\ \text{positive}}} \| x^*x+z \| = \inf_{z\varepsilon I} \| x^*x+z \|$, which completes the proof.

The two-sided closed ideal $K = \{x/f(x^*x) = 0,\ \text{for all } f\varepsilon P_0\}$ of a B*-algebra is said the kernel of the algebra, and a function $N(x)$ on R is defined by $N(x) = \sup_{f\varepsilon P_0} f(x^*x)^{\frac{1}{2}}$.

Theorem 1. The following conditions for a B*-algebra are equivalent:

(1) $P_0 \neq$ empty and $K \neq (0)$, (2) $P_0 \neq$ empty and $N(x) = \| x \|$, for every $x\varepsilon R$, (3) The condition (*c*).

Proof. (3) $\to$ (2), On account of Lemma 5; (2) $\to$ (1), Evident; (1) $\to$ (3), Omitted here.

Theorem 2. Any C*-algebra with the unit is isomorphic and isometric to a self-adjoint algebra of the operators in a Hilbert space, which is closed in the uniform norm topology.

Proof. Omitted (cf. [2]).

Lemma 7. Let I be a closed two-sided ideal in a B*- or C*-algebra R with the unit. Then the quotient algebra R/I is again a B*- or C*-algebra.

Proof. That the quotient algebra R/I, with the norm $\| X \| = \inf_{x\varepsilon X} \| x \|$, $X\varepsilon R/I$, is a normed ring can be easily proved as usually. Lemma 2 shows that the correspondence $x \to x^*$, where x, x^* denotes the residue classes of x, x^* modulo I, satisfies the conditions (a) and (d) or (a), (c) and (d). Further, we can see, on account of Lemma 6, that $\|x\| = \|x^*x\|$, which is the condition (b). This completes the proof.

Semi-Simplicity and Weak Semi-Simplicity. A B*- or C*-algebra with the unit element is called *weakly semi-simple*, if the intersection of all the maximal left ideals is the null ideal and the intersection of all the maximal right ideals is the null ideal.

A B*- or C*-algebra is called *semi-simple* if the intersection of all the maximal two-sided ideals, which is defined to be the radical of the algebra, is the null ideal, and is called *simple* if its only two-sided ideals are the null ideal and the whole algebra.

Lemma 8. For any left closed ideal I in a C*-algebra, there is a $f(x) \varepsilon P_0$ such that $f(x^*x) = 0$, for all $x \varepsilon I$.

Proof. The linear closed subspace H_I does not contain any point of the convex, conical open set D_0. By the well-known theorem of Ascoli-Mazur [3, 4], there is a (real-valued) linear functional $f(x)$ on E_R such that $f(D_0) > 0$, $f(H_I) = 0$ and $f(e) = 1$; consequently it holds that $f(x^*x) \geqq 0$, for all $x \varepsilon R$ and $f(x^*x) = 0$, for every $x \varepsilon I$ (Cf. Lemma 1). $g(x) = f(u) + if(v)$, $x = u + iv \varepsilon R$, is the required functional of P_0.

By this lemma, we can find that, for every maximal left (or right) ideal I of any C*-algebra R, there is a $f(x) \varepsilon P_0$ such that $I = \{x \varepsilon R / f(x^*x) = 0\}$[1], and that the weak semi-simplicity of any C*-algebra is equivalent to the condition $K = (0)$.

More generally, we have the following

Theorem 3. Any closed two-sided ideal I in a C*-algebra is the intersection of all its maximal left ideals each of which contains the ideal or is the intersection of all its maximal right ideals each of which contains the ideal.

Proof. Consider the residue class C*-algebra R/I of R modulo the given two-sided closed ideal I, then R/I is again a C*-algebra, to which we can apply Theorem 1 and the preceeding remark, to obtain the theorem.

Theorem 3 is equivalent to the following theorem 3′ due to Segal:

Theorem 3′. (I.E. Segal) For any two-sided ideal in a C*-algebra, there is a collection of positive linear functionals (or of minimal positive linear functionals) such that the ideal consists of just those elements which vanish in every functional of the collection.

Theorem 4. Every simple B*-algebra with the unit for which we assume that P_0 is not empty is a C*-algebra.

---

1) Cf. Godement, Trans, 1948, Appendix C. This fact has an important application to the theory of unitary representation of the locally compact group. [7, 8]

Proof. This can also simply see as follows: Let $\mathfrak{D}$ be the set $\{u/u\varepsilon E_R$ and $f(u)\geqq 0$, for all $f\varepsilon P_0\}$, then $K=\{x/x^*x\varepsilon\mathfrak{D}\wedge(-\mathfrak{D})\}$, and $K=(0)$ implies that, for every $x\neq 0$, there is a $f(x)\varepsilon P_0$ such that $f(x^*x)>0$.

Theorem 5. The quotient algebra of any B*-algebra with the unit modulo any maximal two-sided ideal is a simple C*-algebra.

Proof. Let R be any B*-algebra with the unit element and I any maximal two-sided ideal of R. Then, the quotient algebra R/I is, by Lemma 7, a B*-algebra. As there is no two-sided closed ideal in R/I, R/I must be a C*-algebra, by Theorem 4.

Corollary. Every maximal two-sided ideal in any B*-algebra is the intersection of all maximal left ideals which contain the ideal; and the same, for maximal right ideals.

Theorem 6. Every semi-simple B*-algebra with the unit element has an isometric isomorphism into the direct product algebra of simple C*-algebras of bounded operators on a Hilbert space.

Proof. We take the totality J_α, $\alpha\varepsilon A$ of all maximal two-sided ideals of a B*-algebra R. Let the quotient algebra R/J_α be isomorphic with the C*-operator-algebra C_α on Hilbert spaces $\mathfrak{H}_\alpha$. Let us define the Hilbert space $\mathfrak{H}$ as follows: the element ξ of $\mathfrak{H}$ is the vector-valued functions defined on the set A: $\xi=\{\xi_\alpha,\ \alpha\varepsilon A\}$, for which $\sup\limits_{\alpha\varepsilon A}\|\xi_\alpha\|<\infty$; the norm being defined by $\|\xi\|=\sup\limits_{\alpha\varepsilon A}\|\xi_\alpha\|$. By the Jordan-von Neumann criterion we can define an inner product $(\xi,\ \eta)$, induced by this norm, and find that $\mathfrak{H}$ is a Hilbert space. For every element x of R, we make correspond the operator $T(x)$ on $\mathfrak{H}$ such that $T(x)\xi=\{T_\alpha(x)\xi_\alpha,\ \alpha\varepsilon A\}$, where $T_\alpha(x)$ denote the image on C_α of the element in R/J corresponding to x. Then it is easily seen that the correspondence $x\rightarrow T(x)$ is a continuous homomorphism of R into the direct product algebra of C*-algebras of operators on Hilbert spaces $\mathfrak{H}_\alpha$.

As it is easily seen $|||T(x)|||=\sup\limits_{\alpha\varepsilon A}|||T_\alpha(x)|||$, and $|||T_\alpha(x)|||=\|x\|=N_\alpha(x)=\sup\limits_{f\varepsilon P_\alpha}f(x^*x)^{\frac{1}{2}}\leqq\|x\|$, for each C_α, and for every $x\varepsilon C_\alpha$, $\alpha\varepsilon A$, where P_α denotes the totality of positive linear functionals $f(x)\varepsilon P_0$ such that $f(J_\alpha)=0$. Therefore $T(x)=0$ if and only if $x\varepsilon\bigwedge\limits_{\alpha\varepsilon A}J_\alpha$.

Corollary. Every semi-simple B*-algebra is weakly semi-simple.

Appendix. We will add some remarks.

(1) Let R be a normed algebra without the unit element such that the involution $x\rightarrow x^*$ is defined and satisfies: (1) $(\alpha x+\beta y)^*=\bar{\alpha}x^*+\bar{\beta}y^*$, (2) $(xy)^*=y^*x^*$, (3) $x^{**}=x$, (4) there is a constant $h>0$ such that $h\|x\|^2\leqq\|x^*x\|$, for all $x\varepsilon R$. Then R can be imbedded into a normed algebra with the unit element

satisfying (1)—(4), with the equivalent norm.

Proof. For every $x \varepsilon R$, we make correspond the operator A_x on the Banach space R such that $A_x \cdot z = xz\,(z \varepsilon R)$, and consider all those bounded operators $\alpha I + A_x$: $(\alpha I + A_x)\, z = \alpha z + xz\,(z \varepsilon R)$, in which we define $(\alpha I + A_x)^* = \bar{\alpha} I + A_x{}^*$. They form a normed algebra $R(I)$ with the unit I, for which (1)—(3) are obviously valid.

By (4) and $\| xy \| \leqq \| x \| \cdot \| y \|$, we have $\| x^* x \| \leqq h^{-1} \cdot \| x \|^2$ and $h \leqq \| x \| / \| x^* \| \leqq h^{-1}$; as $\| I + A_x \|^2 = \sup_{\| z \| \leq 1} \| z + xz \| \sup_{\| z \| \leq 1} \| z + xz \| = \sup_{\| z \| \leq 1} \| z + xz \|^2 \leqq h^{-1}. \sup_{\| z \| \leq 1} \| (z + xz)^* (z + xz) \| \leqq h^{-2}. \sup_{\| z \| \leq 1} \| z + x^* z + xz + x^* x \cdot z \| = h^{-2} \| (I + A_x)^* (I + A_x) \|$, we have $h^2 \| I + A_x \| \leqq \| I + A_{x^*} \|$ and $h^2 \| I + A_{x^*} \| \leqq \| I + A_x \|$; thus, we have $h^2 \| I + A_x \|^2 \leqq \| (I + A_x)^* (I + A_x) \| \leqq h^{-2} \| I + A_x \|^2$, which proves (4).

(2). Let A be a commutative normed algebra with an involution such that the conditions (1)—(4) are valid. Then A is represented isomorphically and isometrically onto the ring of all complex-valued, continuous, bounded functions on the compact Hausdorff space of all maximal ideals of the normed algebra $A(I)$ such that $x\{(A)\} = 0$ and $x^*(M) = \overline{x(M)}$, where (A) denotes the maximal ideals A in $A(I)$.

Proof. We only prove the realness of the spectrum of any hermitian element u in $A(I)$. Let v be the element $v = e^{iu} = \sum_{n=0}^{\infty} \frac{1}{n!} (iu)^n$. As $v^{-1} = v^*$, $v^* v = v^{-1} v = e$, we have $h^2 \| v \|^2 \leqq \| v^* v \| \leqq h^{-2} \| v \|^2$, so we have: $h^{\frac{1}{n}} \leqq \| v^{2n} \|^{\frac{1}{2n}} \leqq h^{-\frac{1}{2n}}$, thus, $\sup_{M \varepsilon \mathfrak{M}} | v(M) | = 1$; in the same manner, we have $\sup_{M \varepsilon \mathfrak{M}} | v^{-1}(M) | = 1$; therefore, $| v(M) | = 1$, for every $M \varepsilon \mathfrak{M}$, which completes the proof.

References.

1) I. E. Segal, Two-Sided Ideals of Operator Algebras, Annals of Math., Vol. 50 (1949).

2) Gelfand and Neumark, On the Imbedding of Normed Rings into the Ring of Operators in Hilbert Spaces, Rec. Math. N. S., Vol. 12 (1943).

3) S. Mazur, Studia Math., Vol. 4 (1932), pp. 73–74.

4) M. Krein, Sur les operations lineares transforment un certain ensemble conique en lui-meme, C.R. URSS, Vol. 23 (1938).

5) I. E. Segal, The Group Algebra of a Locally Compact Group, Trans. Amer. Math. Soc. Vol. 61 (1947).

6) I. Kaplansky, Normed Algebra, Duke Math. Journ. Vol. 16 (1949) No. 3.

7) R. Godement, Trans. Amer. Math. Soc.,Vol. 63 (1948).

8) L. Schwarz, C. R. Paris 227, 424–426 (1948).