72. *The Order of the Derivative of a Meromorphic Function.*

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(Comm. by K. KUNUGI, M.J.A., July 12, 1951.)

The following result is due to Whittaker[1]:

Theorem. *Any meromorphic function is of the same order as its derivative.*

Whittaker's own proof of the theorem was based upon a result concerning the expansion of a meromorphic function into a series of Mittag-Leffler's type which had also been established by himself[2]. He further remarked in the addenda[3] at the end of the Journal containing his paper that Valiron drew his attention to a memoir[4] in which Valiron had previously proved the theorem. But, in the Valiron's paper we can find no detail; in fact, only the following statement is found there:

Signalons encore proposition: *l'ordre ρ d'une fonction méromorphe $f(z)$ et l'ordre de sa dérivée sont égaux.* C'est évident lorsque f est le quotient d'une fonction entière f_1 d'ordre au plus égal à ρ par un produit canonique P d'ordre ρ et dans le cas contraire, la propriété résulte de ce que la fonction $f_1P'-f_1'P$ est d'ordre ρ si f_1 est d'ordre ρ et P d'ordre inférieur à ρ.

Recently, Tsuji has succeeded to give a simple proof of the theorem essentially based upon Valiron's idea which will be in a paper[5] before long published. The last part of the above cited Valiron's statement will really be found in this paper as a lemma accompanied by a proof.

The purpose of the present paper is to give a more brief proof of this interesting theorem. The last part of the Valiron's statement will also be established, as a corollary of the theorem, at the end of the present paper.

Let $f(z)$ be a meromorphic function of order ρ, and let the order of its derivative $f'(z)$ be denoted by ρ'. If $f(z)$ is an integral

1) J. M. Whittaker, The order of the derivative of a meromorphic function. Journ. London Math. Soc. **11** (1936), 82-87.

2) J. M. Whittaker, A theorem on meromorphic function. Proc. London Math. Soc. (2) **40** (1935), 255-272.

3) J. M. Whittaker, Addendum to the previous paper. Journ. London Math. Soc. **11** (1936), 320.

4) G. Valiron, Sur la distribution des valeurs des fonctions méromorphes. Acta Math. **47** (1926), 117-142.

5) M. Tsuji, On the order of the derivative of a meromorphic function. Tôhoku Math. Journ. (2) **3** (1951).

function, the identity $\rho' = \rho$ is almost evident. In fact, either, as noticed in Whittaker's paper, the inequalities

$$\frac{1}{r}(M(r, f)-|f(0)|) \leqq M(r, f') \leqq \frac{1}{r}M(2r, f)$$

can easily be established, where $M(r, F)$ denotes, as usual, the maximum modulus of $F(z)$ on $|z| = r$, whence it follows $\rho' = \rho$ immediately. Or the result may also be deduced from a well-known fact that the order of an integral function $F(z)$ can be expressed in the form

$$\overline{\lim_{n\to\infty}} \frac{n \log n}{\log |c_n|^{-1}},$$

$\{c_n\}$ denoting the Taylor coefficients: $F(z) = \sum c_n z^n$.

Now, in case of general meromorphic function, the function $f(z)$ is expressible as a quotient of two integral functions of order not exceeding ρ. From this it is easily seen that the inequality

$$\rho' \leqq \rho$$

holds good, which is also really a well-known fact. It remains therefore to deduce the opposite inequality

$$\rho' \geqq \rho,$$

which will be proved in the following lines.

The case $\rho' = \infty$ being trivial, it may and so will be supposed that $\rho' < \infty$. The derivative $f'(z)$ can be expressed in the form

$$f'(z) = \frac{\varphi(z)}{\psi(z)},$$

where $\varphi(z)$ and $\psi(z)$ are both integral functions of order not exceeding ρ'. Moreover, one may take as $\psi(z)$ the canonical product composed of the poles $\{z_\nu\}$ of $f'(z)$. Then, a theorem due to Borel[6] asserts that, for arbitrary positive number ε, if about each point z_ν of modulus greater than unity as centre a circle of radius $|z_\nu|^{-\eta}$ with $\eta > \rho'$ is described, so at any point z outside all these circles the inequality

$$\log |\psi(z)| > -r^{\rho'+\varepsilon}$$

6) É. Borel, Sur les zéros des fonctions entières. Acta Math. **20** (1896), 357-396. Cf. also, for instance, G. Valiron, Lectures on the general theory of integral functions. Toulouse (1923), Theorem 19, p. 57; or E. T. Copson, An introduction to the theory of functions of a complex variable. Oxford (1935), p. 173.

holds provided $r=|z|\geqq r_\varepsilon$. By choosing r_ε sufficiently large, one may suppose that the inequality

$$\log|\varphi(z)|<r^{\rho'+\varepsilon}$$

also holds simultaneously. Hence, it follows that the inequality

$$\log|f'(z)|<2r^{\rho'+\varepsilon}$$

holds good outside all the above mentioned circles, provided $r\geqq r_\varepsilon$. On the other hand, the convergence exponent of $\{z_\nu\}$ coinciding with the order of $\psi(z)$, the series

$$\sum_\nu|z_\nu|^{-\eta}$$

converges, since $\eta>\rho'$. Hence, replacing r_ε, if necessary, by a suitably large number, there exists a half-line

$$\arg z=\alpha, \qquad |z|\geqq r_\varepsilon$$

lying outside all the above circles in question. Let further the sum of the circular projections of all these circles on the positive real axis be denoted by

$$\{p_\nu\leqq x\leqq q_\nu\}.$$

It is immediately seen that

$$l\equiv\sum_\nu(q_\nu-p_\nu)\leqq 2\sum_\nu|z_\nu|^{-\eta}<\infty.$$

Let $r\,(\geqq r_\varepsilon)$ be any point on the real axis which does not belong to the set of projections. Then, for a point $z=re^{i\theta}$ it follows that

$$\begin{aligned}|f(z)|&=\left|f(r_\varepsilon e^{i\alpha})+\left(\int_{r_\varepsilon e^{i\alpha}}^{re^{i\alpha}}+\int_{re^{i\alpha}}^{re^{i\theta}}\right)f'(z)dz\right|\\&=O\left(1+\int_{r_\varepsilon}^{r}\exp(2t^{\rho'+\varepsilon})dt+\int_0^{2\pi}\exp(2r^{\rho'+\varepsilon})rd\phi\right)\\&=O(\exp(2r^{\rho'+2\varepsilon})),\end{aligned}$$

O-notations depending on $r\to\infty$; whence it follows that

$$m(r,f)=O(r^{\rho'+2\varepsilon}).$$

Since the poles of $f(z)$ consist of the corresponding ones of $f'(z)$, multiplicity being diminished by one, it is evident that

$$N(r,f)\leqq N(r,f')=O(r^{\rho'+\varepsilon}).$$

Hence, one concludes that

$$T(r, f) = O(r^{\rho'+2\varepsilon})$$

provided r ($\geqq r_\varepsilon$) does not belong to a set on the real axis whose total length is equal to l.

For any remaining value r ($\geqq r_\varepsilon$) there exists a value r' with $r < r' < r+l+1$ which does not belong to the set. The monotone increasing character of the characteristic function $T(r, f)$ —— moreover, $T(r, f)$ is really a convex function of $\log r$ —— implies

$$T(r, f) \leqq T(r', f) = O((r+l+1)^{\rho'+2\varepsilon}) = O(r^{\rho'+2\varepsilon}).$$

Consequently, since ε is an arbitrary positive number, it follows the relation

$$\rho \equiv \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r} \leqq \rho',$$

yielding the desired result.

In conclusion, it will immediately be deduced from the just proved theorem that, if $F(z)$ and $G(z)$ are integral functions of order equal to and less than ρ respectively, then $F'(z)G(z)-F(z)G'(z)$ is of order ρ; the fact which has also be noticed by Whittaker. In fact, otherwise, it follows that the order of the derivative of $F(z)/G(z)$ would become less than ρ what is evidently absurd.

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