# 89. *Determination of a 3-Cohomology Class in an Algebraic Number Field and Belonging Algebra-Classes.*

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Let $k$ be an algebraic number field (of finite degree) and $K/k$ be a (finite) Galois extension with Galois group $\mathfrak{K}$. Let $I_K$, $P_K$ be the groups of idèles and principal idèles in $K$. The class field theory gives rise to a factor set, of $\mathfrak{K}$, in the factor group of the idèle-class group $\mathfrak{C}_K = I_K/P_K$ modulo its component of unity. This factor set can be represented by a certain factor set in the idèle-class group $\mathfrak{C}_K$ itself which satisfies some further requirements, as was shown by Weil [7]; a different, direct derivation of the same factor set has been given in Nakayama [4], [5]. This last factor set in $\mathfrak{C}_K$ is called a canonical factor set for $K/k$, and is determined uniquely by $K/k$ in the sense of equivalence. Let $\{\mathfrak{a}(\sigma, \tau)\}$ $(\sigma, \tau \in \mathfrak{K})$ be a such canonical factor set for $K/k$ and $a(\sigma, \tau)$ be idèles which represent the idèle-classes $\mathfrak{a}(\sigma, \tau)$. Then the coboundary $\alpha = \delta a$ (given by $\alpha(\rho, \sigma, \tau) = a(\sigma, \tau)a(\rho\sigma, \tau)^{-1}a(\rho, \sigma\tau)a(\rho, \sigma)^{-\tau}$) is a 3-cochain in $P_K$ and is in fact a 3-cocycle. In this way we have a 3-cohomology class $\alpha$ in $P_k$ attached in invariant manner to $K/k$. The order of this 3-cohomology class $\alpha$ has been determined in [5] and is equal to the degree $(K:k)$ divided by the least common multiple of $p$-degrees of $K/k$, $p$ running over all primes in $k$.

On the other hand, if $\mathfrak{A}$ is a central simple algebra over $K$ such that every $\sigma \in \mathfrak{K}$ can be extended to an automorphism of $\mathfrak{A}$, then $\mathfrak{A}$ determines a certain 3-cohomology class in $P_K$, called the Teichmüller class of $\mathfrak{A}$ ([6]). MacLane [3] has shown that the totality of the 3-cohomology classes arising in this way (with different $\mathfrak{A}$'s) form a cyclic group of the same order as that of $\alpha$ described above. In fact, it was shown by Hochschild and the writer that $\alpha$ is a generator of this cyclic group ([2]).

Now arises the problem to determine the exact algebra-class (though not unique) whose Teichmüller-class is (not only a power (with exponent prime to the above order) of, but) exactly our $\alpha$, attached invariantly to $K/k$. The answer is given by the following theorem: Let $n_p$ be the $p$-degree of $K/k$, for a prime $p$ in $k$, and let $n'$ be the least common multiple of all the $n_p$, $p$ running over all primes in $k$. Then $\mathfrak{A}$ *has* $\alpha$ *as its Teichmüller-class if, and only* *if*, $\left(\left(\frac{\mathfrak{A}}{P}\right)=\left(\frac{\mathfrak{A}}{P'}\right)\right.$ *for any two primes* $P$, $P'$ *in* $K$ *dividing a same prime* $p$ *in* $k$ *and*$\left.\right)$

$$(1)\qquad \sum_p \left(\frac{\mathfrak{A}}{P}\right)\frac{n'}{n_p} \equiv \frac{-n'}{(L:k)} \quad (mod\ 1)\,,$$

*where* $\left(\frac{\mathfrak{A}}{P}\right)$ *is the Hasse invariant* ([1]) *of the algebra* $\mathfrak{A}$ *for a prime* $P$ *in* $K$ *dividing* $p$.

A particular instance of $\mathfrak{A}$ is the case where $p$ is an arbitrary but fixed prime[1] in $k$ and

$$(2)\qquad \left(\frac{\mathfrak{A}}{P}\right) \equiv \begin{cases} -n_p/(K:k) & P \mid p \\ 0 & P \nmid p\,. \end{cases}$$

Another case of $\mathfrak{A}$, which makes the common denominator of the invariants smallest, is the following one, where

$$(3)\qquad \left(\frac{\mathfrak{A}}{P}\right) \equiv \begin{cases} -t_i n'/(K:k) & \text{for } P \mid p_i \\ 0 & \text{for } P \text{ dividing no } p_i\,, \end{cases}$$

with $1/n' = \sum t_i/n_{p_i}$.

For the proof, we take an auxiliary cyclic field $Z$ over $k$ such that $(Z:k)=(K:k)$ and $Z \frown K = k$. Let $\mathfrak{Z}$ be the Galois group of $Z/k$ and $\tau$ be a generator of $\mathfrak{Z}$. Take a prime[2] idèle $p$ in $k$ whose Frobenius-Artin-Chevalley-symbol $(p, K/k)$ is $\tau$. Now, since $(p, KZ/K)=1$, there exist a $\alpha$ in $P_k$ and an $A$ in $I_{KZ}$ such that

$$(4)\qquad p = \alpha N_{KZ/K}(A)\,.$$

We consider the cyclic algebra $\mathfrak{A} = (\alpha^{-1}, KZ, \tau)$, which is a central simple algebra of degree $(K:k)=(Z:k)$ over $K$. We see from (4) that $\mathfrak{A}$ satisfies (2). Every automorphism of $K/k$ can be extended to $\mathfrak{A}$, and $\mathfrak{A}$ determines its Teichmüller 3-cohomology class in $P_k$. The construction of this last can be carried out rather explicitly by means of (4) and the Hilbert-Speiser theorem.

Consider, on the other hand, the (normalized) cyclic (idèle-) factor set defined by our $p$ and $\tau$, which we shall denote simply by $p$, for the sake of brevity. The idèle-class $\mathfrak{p}$ of (the idèle) $p$ defines similarly, with respect to $\tau$, a cyclic (idèle-class) factor set, which we denote again simply by $\mathfrak{p}$. This $\mathfrak{p}$ is in fact a canonical factor set for $Z/k$. The lift $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$ to $KZ/k$ (i.e. to $\mathfrak{K}\times\mathfrak{Z}$) is then equivalent to the $(K:k)$-th power of the canonical factor set for $KZ/k$. The restriction of $\tilde{\mathfrak{p}}$ to $\mathfrak{Z}$, considered as the Galois group of $KZ/K$, is $\sim 1$. Thus there exists a factor set $\mathfrak{a}$ of $\mathfrak{K}$, the Galois

1) Finite, unless $(K:k)=2$ (or 1).

(2Merely for the sake of simplicity of description.

group of $K/k$, in $\mathfrak{C}_K$, such that its lift $\tilde{\mathfrak{a}}$ to $KZ/k$ is equivalent to $\tilde{\mathfrak{p}} : \tilde{\mathfrak{a}} \sim \tilde{\mathfrak{p}}$. $\mathfrak{a}$ is then a canonical factor set for $K/k$; observe that $(Z:k) = (K:k)$. This is in fact the way we derived a canonical factor set for $K/k$ in [5]. The construction of $\mathfrak{a}$ can be carried out by a certain technique in cohomology theory, and the arithmetical back-ground for the construction is given by the idèle interpretation of Noether's principal genus theorem. However, in order to obtain our 3-cocycle $\alpha$ (in $P_K$) belonging to $K/k$, we have to perform this construction in terms of idèles, rather than of idèle-classes, preserving thus all principal idèle factors which appear in several steps of construction. For instance, the application of the principal genus theorem is resolved into Hasse's norm theorem, the Hilbert-Speiser theorem and its idèle analogy.

Thus both the Teichmüller-cocycle of $\mathfrak{A}$ and the 3-cocycle $\alpha$ belonging to $K/k$ can be constructed by means of the mentioned theorems. But these theorems themselves are of abstract nature, in a sense, and it is rather difficult to carry out our construction concretely, in either case. It turns out, however, that there exists a rather remarkable parallelism between two constructions of ours, although the construction of $\alpha$ is far more complicated than that of the Teichmüller-cocycle, and this parallelism ends up in showing that our $\mathfrak{A}$ has $\alpha$ as its Teichmüller-class. Our $\mathfrak{A}$, given by (2), is an instance of the algebra-classes given in our theorem, and our theorem in its general form follows easily from this particular case $\mathfrak{A}$.

The detailed accounts of the above constructions and the discussion of the parallelism will be given in a subsequent paper. It seems to the writer that this 3-cohomology class $\alpha$ and our parallelism in construction are of some significance both for the theory of idèle-classes and for the theory of algebras.

## References.


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